Random Variables and Product of Probability Spaces

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Summary. We have been working on the formalization of the probability and the randomness. In [15] and [16], we formalized some theorems concerning the real-valued random variables and the product of two probability spaces. In this article, we present the generalized formalization of [15] and [16]. First, we formalize the random variables of arbitrary set and prove the equivalence between random variable on Σ, Borel sets and a real-valued random variable on Σ. Next, we formalize the product of countably infinite probability spaces.

The notation and terminology used in this paper have been introduced in the following articles: [1], [14], [12], [4], [11], [18], [7], [8], [5], [2], [3], [9], [13], [22], [15], [16], [20], [21], [17], [19], [6], and [10].

1. Random Variables

In this paper Ω, Ω₁, Ω₂ denote non empty sets, Σ denotes a σ-field of subsets of Ω, S₁ denotes a σ-field of subsets of Ω₁, and S₂ denotes a σ-field of subsets of Ω₂.

Now we state the proposition:

(1) Let us consider a non empty set B and a function f. Then \( f^{-1}(\bigcup B) = \bigcup \{ f^{-1}(Y) \mid Y \text{ is an element of } B : \text{not contradiction} \} \).

Let us consider a function f from Ω₁ into Ω₂, a sequence B of subsets of Ω₂, and a sequence D of subsets of Ω₁. Now we state the propositions:

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If for every element $n$ of $\mathbb{N}$, $D(n) = f^{-1}(B(n))$, then $f^{-1}(\bigcup B) = \bigcup D$.

If for every element $n$ of $\mathbb{N}$, $D(n) = f^{-1}(B(n))$, then $f^{-1}(\text{Intersection } B) = \text{Intersection } D$.

Now we state the propositions:

(4) Let us consider a function $F$ from $\Omega$ into $\mathbb{R}$ and a real number $r$. Suppose $F$ is a real-valued random variable on $\Sigma$. Then $f^{-1}([-\infty, r[) \in \Sigma$.

Proof: Consider $X$ being an element of $\Sigma$ such that $X = \Omega$ and $F$ is measurable on $X$. For every element $z$, $z \in f^{-1}([-\infty, r[)$ if and only if $z \in \Omega \cap \text{LE-dom}(F, r)$.

□

(5) Let us consider a function $F$ from $\Omega$ into $\mathbb{R}$. Suppose $F$ is a real-valued random variable on $\Sigma$. Then $\{x \mid x \text{ is an element of the Borel sets: } f^{-1}(x) \text{ is element of } \Sigma\}$ is the Borel sets.

Theorem: Let us consider a function $f$ from $\Omega$ into $\mathbb{R}$. Suppose $f$ is a real-valued random variable on $\Sigma$. Then $f$ is random variable of $\Sigma$ and the Borel sets if and only if $f$ is a real-valued random variable on $\Sigma$.

□

(6) Suppose $f$ is a real-valued random variable on $\Sigma$. Then $\{x \mid x \text{ is an element of the Borel sets: } f^{-1}(x) \text{ is element of } \Sigma\}$ is the Borel sets.

(7) $f$ is random variable on $\Sigma$ and the Borel sets if and only if $f$ is a real-valued random variable on $\Sigma$.

(8) The set of random variables on $\Sigma$ and the Borel sets = the real-valued random variables set on $\Sigma$.

Let us consider $\Omega_1$, $\Omega_2$, $S_1$, and $S_2$. Let $F$ be a function from $\Omega_1$ into $\Omega_2$. We say that $F$ is $(S_1, S_2)$-random variable-like if and only if

(Def. 1) $F$ is random variable on $S_1$ and $S_2$.

Observe that there exists a function from $\Omega_1$ into $\Omega_2$ which is $(S_1, S_2)$-random variable-like.

A random variable of $S_1$ and $S_2$ is an $(S_1, S_2)$-random variable-like function from $\Omega_1$ into $\Omega_2$. Now we state the proposition:

(9) Let us consider a function $f$ from $\Omega$ into $\mathbb{R}$. Then $f$ is a random variable of $\Sigma$ and the Borel sets if and only if $f$ is a real-valued random variable on $\Sigma$.

Let $F$ be a function. We say that $F$ is random variable family-like if and only if

(Def. 2) Let us consider a set $x$. Suppose $x \in \text{dom } F$. Then there exist non empty sets $\Omega_1$, $\Omega_2$ and there exists a $\sigma$-field $S_1$ of subsets of $\Omega_1$ and there exists
a σ-field $S_2$ of subsets of $\Omega_2$ and there exists a random variable $f$ of $S_1$ and $S_2$ such that $F(x) = f$.

One can verify that there exists a function which is random variable family-like.

A random variable family is a random variable family-like function. In this paper $F$ denotes a random variable of $S_1$ and $S_2$.

Let $Y$ be a non empty set, $S$ be a σ-field of subsets of $Y$, and $F$ be a function.

We say that $F$ is $S$-measure valued if and only if

(Def. 3) Let us consider a set $x$. If $x \in \text{dom } F$, then there exists a σ-measure $M$ on $S$ such that $F(x) = M$.

Note that there exists a function which is $S$-measure valued.

Let $F$ be a function. We say that $F$ is $S$-probability valued if and only if

(Def. 4) Let us consider a set $x$. If $x \in \text{dom } F$, then there exists a probability $P$ on $S$ such that $F(x) = P$.

Let us note that there exists a function which is $S$-probability valued.

Let $X, Y$ be non empty sets. One can verify that there exists an $S$-probability valued function which is $X$-defined.

One can verify that there exists an $X$-defined $S$-probability valued function which is total.

Let $Y$ be a non empty set. Let us note that every function which is $S$-probability valued is also $S$-measure valued.

Let $F$ be a function. We say that $F$ is $S$-random variable family if and only if

(Def. 5) Let us consider a set $x$. Suppose $x \in \text{dom } F$. Then there exists a real-valued random variable $Z$ on $S$ such that $F(x) = Z$.

Observe that there exists a function which is $S$-random variable family. 

Now we state the propositions:

(10) Let us consider an element $y$ of $S_2$. Suppose $y \neq \emptyset$. Then \{ \{z \text{ where } z \text{ is an element of } \Omega_1 : F(z) \text{ is an element of } y\} = F^{-1}(y)\). \text{Proof:} Set $D = \{z \text{ where } z \text{ is an element of } \Omega_1 : F(z) \text{ is an element of } y\}$. For every element $x, x \in D$ iff $x \in F^{-1}(y)$. \(\square\)

(11) Let us consider a random variable $F$ of $S_1$ and $S_2$. Then

(i) \{ \{x \text{ where } x \text{ is a subset of } \Omega_1 : \text{ there exists an element } y \text{ of } S_2 \text{ such that } x = F^{-1}(y)\} \subseteq S_1\), and

(ii) \{ \{x \text{ where } x \text{ is a subset of } \Omega_1 : \text{ there exists an element } y \text{ of } S_2 \text{ such that } x = F^{-1}(y)\} \text{ is a } \sigma\text{-field of subsets of } \Omega_1\).

The theorem is a consequence of (3). \text{Proof:} Set $S = \{x \text{ where } x \text{ is a subset of } \Omega_1 : \text{ there exists an element } y \text{ of } S_2 \text{ such that } x = F^{-1}(y)\}$.

For every element $x$ such that $x \in S$ holds $x \in S_1$. For every subset $A$ of
\( \Omega_1 \) such that \( A \in S \) holds \( A^c \in S \). For every sequence \( A_1 \) of subsets of \( \Omega_1 \) such that \( \text{rng} \ A_1 \subseteq S \) holds Interception \( A_1 \in S \). □

Let us consider \( \Omega_1, \Omega_2, S_1, \) and \( S_2 \). Let \( M \) be a measure on \( S_1 \) and \( F \) be a random variable of \( S_1 \) and \( S_2 \). The functor the image measure of \( F \) and \( M \) yielding a measure on \( S_2 \) is defined by

(Def. 6) Let us consider an element \( y \) of \( S_2 \). Then \( it(y) = M(F^{-1}(y)) \).

Let \( M \) be a \( \sigma \)-measure on \( S_1 \). Note that the image measure of \( F \) and \( M \) is \( \sigma \)-additive.

Now we state the proposition:

(12) Let us consider a probability \( P \) on \( S_1 \) and a random variable \( F \) of \( S_1 \) and \( S_2 \). Then \( \text{image measure of } F \text{ and } P \) \( \text{on } S_2 \) is defined by the term

(Def. 7) \( M \) the image measure of \( F \) and \( P \).

Now we state the propositions:

(13) Let us consider a probability \( P \) on \( S_1 \) and a random variable \( F \) of \( S_1 \) and \( S_2 \). Then \( \text{image measure of } F \text{ and } P \) \( \text{on } S_2 \) is defined by the term

(14) Let us consider a probability \( P \) on \( S_1 \), a random variable \( F \) of \( S_1 \) and \( S_2 \), and a set \( y \). If \( y \in S_2 \), then \( \text{image measure of } F \text{ and } P \) \( \text{on } S_2 \) is defined by the term

(15) Every function from \( \Omega_1 \) into \( \Omega_2 \) is a random variable of the trivial \( \sigma \)-field of \( \Omega_1 \) and the trivial \( \sigma \)-field of \( \Omega_2 \).

(16) Let us consider a non empty set \( S \). Then every non empty finite sequence of elements of \( S \) is a random variable of the trivial \( \sigma \)-field of \( S \) and the trivial \( \sigma \)-field of \( S \).

(17) Let us consider finite non empty sets \( V, S \), a random variable \( G \) of the trivial \( \sigma \)-field of \( V \) and the trivial \( \sigma \)-field of \( S \), and a set \( y \). Suppose \( y \in \text{the trivial } \sigma \text{-field of } S \). Then \( \text{image measure of } G \text{ and } \text{the trivial probability of } V \) \( \text{on } S \) is defined by the term

(18) Let us consider a finite non empty set \( S \), a non empty finite sequence \( s \) of elements of \( S \), and a set \( x \). Suppose \( x \in S \). Then there exists a random variable \( G \) of the trivial \( \sigma \)-field of \( S \) such that

(i) \( G = s \), and

(ii) \( \text{image measure of } G \text{ and } \text{the trivial probability of } S \) \( \text{on } \{x\} \) = \( \text{Prob}_D(x, s) \).

The theorem is a consequence of (16) and (17).
2. Product of Probability Spaces

Let $D$ be a non-empty many sorted set indexed by $\mathbb{N}$ and $n$ be a natural number. One can check that $D(n)$ is non empty.

Let $S$, $F$ be many sorted sets indexed by $\mathbb{N}$. We say that $F$ is $\sigma$-field $S$-sequence-like if and only if

(Def. 8) Let us consider a natural number $n$. Then $F(n)$ is a $\sigma$-field of subsets of $S(n)$.

Let $S$ be a many sorted set indexed by $\mathbb{N}$. Let us observe that there exists a many sorted set indexed by $\mathbb{N}$ which is $\sigma$-field $S$-sequence-like.

Let $D$ be a many sorted set indexed by $\mathbb{N}$. A $\sigma$-field sequence of $D$ is a $\sigma$-field $D$-sequence-like many sorted set indexed by $\mathbb{N}$. Let $S$ be a $\sigma$-field sequence of $D$ and $n$ be a natural number. Note that the functor $S(n)$ yields a $\sigma$-field of subsets of $D(n)$. Let $D$ be a non-empty many sorted set indexed by $\mathbb{N}$. Let $M$ be a many sorted set indexed by $\mathbb{N}$. We say that $M$ is $S$-probability sequence-like if and only if

(Def. 9) Let us consider a natural number $n$. Then $M(n)$ is a probability on $S(n)$.

Observe that there exists a many sorted set indexed by $\mathbb{N}$ which is $S$-probability sequence-like.

A probability sequence of $S$ is an $S$-probability sequence-like many sorted set indexed by $\mathbb{N}$. Let $P$ be a probability sequence of $S$ and $n$ be a natural number. One can verify that the functor $P(n)$ yields a probability on $S(n)$. Let $D$ be a many sorted set indexed by $\mathbb{N}$. The functor the product domain $D$ yielding a many sorted set indexed by $\mathbb{N}$ is defined by

(Def. 10) (i) $it(0) = D(0)$, and
(ii) for every natural number $i$, $it(i + 1) = it(i) \times D(i + 1)$.

Now we state the proposition:

(19) Let us consider a many sorted set $D$ indexed by $\mathbb{N}$. Then
(i) (the product domain $D)(0) = D(0)$, and
(ii) (the product domain $D)(1) = D(0) \times D(1)$, and
(iii) (the product domain $D)(2) = D(0) \times D(1) \times D(2)$, and
(iv) (the product domain $D)(3) = D(0) \times D(1) \times D(2) \times D(3)$.

Let $D$ be a non-empty many sorted set indexed by $\mathbb{N}$. Let us note that the product domain $D$ is non-empty.

Let $D$ be a finite-yielding many sorted set indexed by $\mathbb{N}$. One can check that the product domain $D$ is finite-yielding.

Let us consider $\Omega$ and $\Sigma$. Let $P$ be a set. Assume $P$ is a probability on $\Sigma$.

The functor modetrans($P$, $\Sigma$) yielding a probability on $\Sigma$ is defined by the term

(Def. 11) $P$. 
Let $D$ be a finite-yielding non-empty many sorted set indexed by $\mathbb{N}$. The functor the trivial $\sigma$-field sequence $D$ yielding a $\sigma$-field sequence of $D$ is defined by

(Def. 12) Let us consider a natural number $n$. Then $it(n) =$ the trivial $\sigma$-field of $D(n)$.

Let $P$ be a probability sequence of the trivial $\sigma$-field sequence $D$ and $n$ be a natural number. One can check that the functor $P(n)$ yields a probability on the trivial $\sigma$-field of $D(n)$. The functor $\text{ProductProbability}(P,D)$ yielding a many sorted set indexed by $\mathbb{N}$ is defined by

(Def. 13) (i) $it(0) = P(0)$, and

(ii) for every natural number $i$, $it(i + 1) = \text{Product-Probability}((\text{the product domain } D)(i), D(i + 1), \text{modetrans} (it(i), \text{the trivial } \sigma\text{-field of } (\text{the product domain } D)(i)), P(i + 1))$.

Let us consider a finite-yielding non-empty many sorted set $D$ indexed by $\mathbb{N}$, a probability sequence $P$ of the trivial $\sigma$-field sequence $D$, and a natural number $n$. Now we state the propositions:

(20) $(\text{ProductProbability}(P,D))(n)$ is a probability on the trivial $\sigma$-field of $(\text{the product domain } D)(n)$.

(21) There exists a probability $P_4$ on the trivial $\sigma$-field of $(\text{the product domain } D)(n)$ such that

(i) $P_4 = (\text{ProductProbability}(P,D))(n)$, and

(ii) $(\text{ProductProbability}(P,D))(n+1) = \text{Product-Probability}((\text{the product domain } D)(n), D(n + 1), P_4, P(n + 1))$.

Now we state the proposition:

(22) Let us consider a finite-yielding non-empty many sorted set $D$ indexed by $\mathbb{N}$ and a probability sequence $P$ of the trivial $\sigma$-field sequence $D$. Then

(i) $(\text{ProductProbability}(P,D))(0) = P(0)$, and

(ii) $(\text{ProductProbability}(P,D))(1) = \text{Product-Probability}(D(0), D(1), P(0), P(1))$, and

(iii) there exists a probability $P_1$ on the trivial $\sigma$-field of $D(0) \times D(1) \times D(2)$ such that $P_1 = (\text{ProductProbability}(P,D))(1)$ and $(\text{ProductProbability}(P,D))(2) = \text{Product-Probability}(D(0) \times D(1), D(2), P_1, P(2))$, and

(iv) there exists a probability $P_2$ on the trivial $\sigma$-field of $D(0) \times D(1) \times D(2)$ such that $P_2 = (\text{ProductProbability}(P,D))(2)$ and $(\text{ProductProbability}(P,D))(3) = \text{Product-Probability}(D(0) \times D(1) \times D(2), D(3), P_2, P(3))$, and

(v) there exists a probability $P_3$ on the trivial $\sigma$-field of $D(0) \times D(1) \times D(2) \times D(3)$ such that $P_3 = (\text{ProductProbability}(P,D))(3)$ and
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(ProductProbability($P, D$))(4) = Product-Probability($D(0) \times D(1) \times D(2) \times D(3), D(4), P_3, P(4)$).

The theorem is a consequence of (19) and (21).

References


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