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Morley’s Trisector Theorem

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Summary. Morley’s trisector theorem states that “The points of intersection of the adjacent trisectors of the angles of any triangle are the vertices of an equilateral triangle” [10].

There are many proofs of Morley’s trisector theorem [12, 16, 9, 13, 8, 20, 3, 18]. We follow the proof given by A. Letac in [15].

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The notation and terminology used in this paper have been introduced in the following articles: [1], [11], [7], [14], [19], [2], [4], [23], [5], [24], [21], [22], and [6].

1. Preliminaries

From now on A, B, C, D, E, F, G denote points of \( E^2_T \).

Now we state the propositions:

1. \( \angle(A, B, A) = 0 \).
2. \( 0 \leq \angle(A, B, C) < 2 \cdot \pi \).
3. (i) \( 0 \leq \angle(A, B, C) < \pi \), or
   (ii) \( \angle(A, B, C) = \pi \), or
   (iii) \( \pi < \angle(A, B, C) < 2 \cdot \pi \).

The theorem is a consequence of (2).

4. \( |F - E|^2 = |A - E|^2 + |A - F|^2 - 2 \cdot |A - E| \cdot |A - F| \cdot \cos \angle(E, A, F) \).
5. If \( A, B, C \) are mutually different and \( 0 < \angle(A, B, C) < \pi \), then \( 0 < \angle(B, C, A) < \pi \) and \( 0 < \angle(C, A, B) < \pi \).
(6) Suppose \( A, B, C \) are mutually different and \( \angle(A, B, C) = 0 \). Then
   (i) \( \angle(B, C, A) = 0 \) and \( \angle(C, A, B) = \pi \), or
   (ii) \( \angle(B, C, A) = \pi \) and \( \angle(C, A, B) = 0 \) and \( \angle(A, B, C) + \angle(B, C, A) + \angle(C, A, B) = \pi \).

(7) Suppose \( A, B, C \) are mutually different and \( \angle(A, B, C) = \pi \). Then
   (i) \( \angle(B, C, A) = 0 \), and
   (ii) \( \angle(C, A, B) = 0 \), and
   (iii) \( \angle(A, B, C) + \angle(B, C, A) + \angle(C, A, B) = \pi \).

(8) If \( A, B, C \) are mutually different and \( \angle(A, B, C) > \pi \), then \( \angle(A, B, C) + \angle(B, C, A) + \angle(C, A, B) = 5 \cdot \pi \).

   Let us assume that \( \angle(C, B, A) < \pi \). Now we state the propositions:
   (9) \( 0 \leq \text{area of } \triangle(A, B, C) \). The theorem is a consequence of (2).
   (10) \( 0 \leq \varphi_0(A, B, C) \). The theorem is a consequence of (9).

2. Morley’s Theorem

Now we state the propositions:

(11) Suppose \( A, F, C \) form a triangle and \( \angle(C, F, A) < \pi \) and \( \angle(A, C, F) = \angle(A, C, B)/3 \) and \( \angle(F, A, C) = \angle(B, A, C)/3 \) and \( \angle(A, C, B)/3 + (\angle(B, A, C)/3) + (\angle(C, B, A)/3) = \pi/3 \). Then \( |A - F| \cdot \sin((\pi/3) - (\angle(C, B, A)/3)) = |A - C| \cdot \sin(\angle(A, C, B)/3) \).

(12) Suppose \( A, B, C \) form a triangle and \( A, F, C \) form a triangle and \( \angle(C, F, A) < \pi \) and \( \angle(A, C, F) = \angle(A, C, B)/3 \) and \( \angle(F, A, C) = \angle(B, A, C)/3 \) and \( \angle(A, C, B)/3 + (\angle(B, A, C)/3) + (\angle(C, B, A)/3) = \pi/3 \) and \( \sin((\pi/3) - (\angle(C, B, A)/3)) \neq 0 \). Then \( |A - F| = 4 \cdot \varphi_0(A, B, C) \cdot \sin(\angle(C, B, A)/3) \cdot \sin((\pi/3) + (\angle(C, B, A)/3)) \cdot \sin(\angle(A, C, B)/3) \). The theorem is a consequence of (11).

(13) Suppose \( C, A, B \) form a triangle and \( A, F, C \) form a triangle and \( A, E \) form a triangle and \( E, A, B \) form a triangle and \( \angle(B, A, E) = \angle(B, A, C)/3 \) and \( \angle(F, A, C) = \angle(B, A, C)/3 \). Then \( \angle(E, A, F) = \angle(B, A, C)/3 \). PROOF: \( \angle(E, A, F) \neq 4 \cdot \pi + (\angle(B, A, C)/3) \) by [17] (5), (2), [7] (30)]. \( \angle(E, A, F) \neq 2 \cdot \pi + (\angle(B, A, C)/3) \) by (2), [7] (30)]. □

(14) Suppose \( C, A, B \) form a triangle and \( \angle(A, C, B) < \pi \) and \( A, F, C \) form a triangle and \( A, E \) form a triangle and \( E, A, B \) form a triangle and \( \angle(B, A, E) = \angle(B, A, C)/3 \) and \( \angle(F, A, C) = \angle(B, A, C)/3 \). Then \( (\pi/3) + (\angle(A, C, B)/3) + ((\pi/3) + (\angle(C, B, A)/3)) + \angle(E, A, F) = \pi \). The theorem is a consequence of (13).
(15) If $A, C, B$ form a triangle, then $\sin((\pi/3) - (\angle(A, C, B)/3)) \neq 0$. The theorem is a consequence of (2).

(16) Suppose $A, B, C$ form a triangle and $A, B, E$ form a triangle and $\angle(E, B, A) = \angle(C, B, A)/3$ and $\angle(B, A, E) = \angle(B, A, C)/3$ and $A, F, C$ form a triangle and $\angle(A, C, F) = \angle(A, C, B)/3$ and $\angle(F, A, C) = \angle(B, A, C)/3$ and $\angle(A, C, B) < \pi$. Then $|F - E| = 4 \cdot \varnothing \cdot \sin(\angle(A, C, B)/3) \cdot \sin(\angle(C, B, A)/3) \cdot \sin(\angle(A, C, B)/3)$.

PROOF: $\sin((\pi/3) - (\angle(A, C, B)/3)) \neq 0$. $\sin((\pi/3) - (\angle(C, B, A)/3)) \neq 0$. $0 < \angle(A, C, B)$. $\angle(C, B, A) < \pi$. $0 < \angle(A, C, B) < \pi$ and $A, C, B$ are mutually different. $\angle(B, A, C) < \pi$. $0 < \angle(B, A, E) < \pi$. $\angle(A, E, B) < \pi$. $0 < \angle(F, A, C) < \pi$. $\angle(C, F, A) < \pi$. $F, A, E$ form a triangle by (4), (5), (17), (5), (7), (31). $|A - F| = \varnothing \cdot (A, B, C) \cdot 4 \cdot \sin(\angle(C, B, A)/3) \cdot \sin((\pi/3) + (\angle(C, B, A)/3)) \cdot \sin(\angle(A, C, B)/3)$. $(\pi/3) + (\angle(A, C, B)/3) + ((\pi/3) + (\angle(C, B, A)/3)) + \angle(E, A, F) = \pi$. $|F - E|^2 = |A - E|^2 + |A - F|^2 - 2 \cdot |A - E| \cdot |A - F| \cdot \cos \angle(E, A, F)$. □

(17) Suppose $A, B, C$ form a triangle and $\angle(E, B, A) = \angle(C, B, A)/3$ and $\angle(B, A, E) = \angle(B, A, C)/3$. Then $A, B, E$ form a triangle. The theorem is a consequence of (1) and (2).

(18) Suppose $A, B, C$ form a triangle and $\angle(A, C, F) = \angle(A, C, B)/3$ and $\angle(F, A, C) = \angle(B, A, C)/3$. Then $A, F, C$ form a triangle. The theorem is a consequence of (1) and (2).

(19) Suppose $A, B, C$ form a triangle and $\angle(C, B, G) = \angle(C, B, A)/3$ and $\angle(G, C, B) = \angle(A, C, B)/3$. Then $C, G, B$ form a triangle. The theorem is a consequence of (1) and (2).

Let us assume that $A, B, C$ form a triangle and $\angle(A, C, B) < \pi$ and $\angle(E, B, A) = \angle(C, B, A)/3$ and $\angle(B, A, E) = \angle(B, A, C)/3$ and $\angle(A, C, F) = \angle(A, C, B)/3$ and $\angle(F, A, C) = \angle(B, A, C)/3$ and $\angle(C, B, G) = \angle(C, B, A)/3$ and $\angle(G, C, B) = \angle(A, C, B)/3$. Now we state the propositions:

(20) (i) $|F - E| = 4 \cdot \varnothing \cdot \sin(\angle(A, C, B)/3) \cdot \sin(\angle(C, B, A)/3)$, and

(ii) $|G - F| = 4 \cdot \varnothing \cdot \sin(\angle(C, B, A)/3)$, and

(iii) $|E - G| = 4 \cdot \varnothing \cdot \sin(\angle(C, B, A)/3)$.

The theorem is a consequence of (17), (18), (19), (2), (5), and (16).

(21) (i) $|F - E| = |G - F|$, and

(ii) $|F - E| = |E - G|$, and

(iii) $|G - F| = |E - G|$.
The theorem is a consequence of (20).

(22) **Morley’s Trisector Theorem:**
Suppose $A$, $B$, $C$ form a triangle and $\angle(A, B, C) < \pi$ and $\angle(E, C, A) = \angle(B, C, A)/3$ and $\angle(C, A, E) = \angle(C, A, B)/3$ and $\angle(A, B, F) = \angle(A, B, C)/3$ and $\angle(F, A, B) = \angle(C, A, B)/3$ and $\angle(B, C, G) = \angle(B, C, A)/3$ and $\angle(G, B, C) = \angle(A, B, C)/3$. Then

(i) $|F - E| = |G - F|$, and

(ii) $|F - E| = |E - G|$, and

(iii) $|G - F| = |E - G|$. The theorem is a consequence of (21).

**References**


Morley’s trisector theorem


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Summary. In this article we introduce necessary notation and definitions to prove the Euler’s Partition Theorem according to H.S. Wilf’s lecture notes [31]. Our aim is to create an environment which allows to formalize the theorem in a way that is as similar as possible to the original informal proof.

Euler’s Partition Theorem is listed as item #45 from the “Formalizing 100 Theorems” list maintained by Freek Wiedijk at http://www.cs.ru.nl/F.Wiedijk/100/ [30].

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The notation and terminology used in this paper have been introduced in the following articles: [1], [2], [6], [8], [15], [27], [13], [11], [23], [9], [10], [7], [25], [24], [3], [4], [19], [5], [22], [32], [33], [11], [21], [28], [18], and [12].

1. Auxiliary Facts about Finite Sequences Concatenation

From now on $x$, $y$ denote objects, $D$, $D_1$, $D_2$ denote non empty sets, $i$, $j$, $k$, $m$, $n$ denote natural numbers, $f$, $g$ denote finite sequences of elements of $D^*$, $f_1$ denotes a finite sequence of elements of $D_1^*$, and $f_2$ denotes a finite sequence of elements of $D_2^*$.
Let us consider a function yielding function $F$, and an object $a$. Then $a \in \text{Values } F$ if and only if there exists $x$ and there exists $y$ such that $x \in \text{dom } F$ and $y \in \text{dom}(F(x))$ and $a = F(x)(y)$.

Let us consider a set $D$, and finite sequences $f$, $g$ of elements of $D^*$. Then $\text{Values } f \upharpoonright g = \text{Values } f \cup \text{Values } g$.

Proof: Set $F = f \upharpoonright g$. $\text{Values } f \subseteq \text{Values } F$ by (1), [6] (26)]. $\text{Values } g \subseteq \text{Values } F$ by (1), [6] (28)]. $\text{Values } F \subseteq \text{Values } f \cup \text{Values } g$ by (1), [6] (25)]. □

The concatenation of $D \circ f \upharpoonright g = (\text{the concatenation of } D \circ f) \upharpoonright (\text{the concatenation of } D \circ g)$.

rng(\text{the concatenation of } D \circ f) = \text{Values } f$.

Proof: Set $D_3 = \text{the concatenation of } D$. Define $P[\text{natural number}] \equiv$ for every finite sequence $f$ of elements of $D^*$ such that $\text{len } f = \$_1$ holds $\text{rng}(D_3 \circ f) = \text{Values } f$. $P[0]$. If $P[i]$, then $P[i + 1]$ by [8] (19), (16)], (3), [27] (11)]. $P[i]$ from [4] Sch. 2]. □

If $f_1 = f_2$, then the concatenation of $D_1 \circ f_1 = \text{the concatenation of } D_2 \circ f_2$.

Proof: Set $C = \text{the concatenation of } D_2$. Set $N = \text{the concatenation of } D_1$. Define $P[\text{natural number}] \equiv$ for every finite sequence $f_4$ of elements of $D_1^*$ for every finite sequence $f_3$ of elements of $D_2^*$ such that $\$_1 = \text{len } f_4$ and $f_4 = f_3$ holds $N \circ f_4 = C \circ f_3$. $P[0]$. If $P[i]$, then $P[i + 1]$ by [8] (19), (16)], (3), [27] (11)]. $P[i]$ from [4] Sch. 2]. □

If $i \in \text{dom}(\text{the concatenation of } D \circ f)$ if and only if there exists $n$ and there exists $k$ such that $n + 1 \in \text{dom } f$ and $k \in \text{dom}(f(n + 1))$ and $i = k + \text{len}(\text{the concatenation of } D \circ f[n])$.

Proof: Set $D_3 = \text{the concatenation of } D$. Define $P[\text{natural number}] \equiv$ for every $i$ for every finite sequence $f$ of elements of $D^*$ such that $\text{len } f = \$_1$ holds $i \in \text{dom}(D_3 \circ f)$ iff there exists $n$ and there exists $k$ such that $n + 1 \in \text{dom } f$ and $k \in \text{dom}(f(n + 1))$ and $i = k + \text{len}(D_3 \circ f[n])$. $P[0]$. If $P[j]$, then $P[j + 1]$ by [8] (19), (16)], (3), [27] (11)]. $P[j]$ from [4] Sch. 2]. □

Suppose $i \in \text{dom}(\text{the concatenation of } D \circ f)$. Then

(i) $(\text{the concatenation of } D \circ f)(i) = (\text{the concatenation of } D \circ f \upharpoonright g)(i)$, and

(ii) $(\text{the concatenation of } D \circ f)(i) = (\text{the concatenation of } D \circ g \upharpoonright f)(i + \text{len}(\text{the concatenation of } D \circ g))$.

The theorem is a consequence of (3).

Suppose $k \in \text{dom}(f(n + 1))$. Then $f(n + 1)(k) = (\text{the concatenation of }$
$D \odot f)(k + \text{len}(\text{the concatenation of } D \odot f[n]))$. The theorem is a consequence of (3).

2. Flexary Plus

From now on $f$ denotes a complex-valued function and $g, h$ denote complex-valued finite sequences.

Let us consider $k$ and $n$. Let $f, g$ be complex-valued functions. The functor $(f, k) + \ldots + (g, n)$ yielding a complex number is defined by

(Def. 1) \[
\begin{align*}
(i) & \quad h(0 + 1) = f(0 + k) \text{ and } \ldots \text{ and } h(n -' k + 1) = f(n -' k + k), \text{ if } f = g \text{ and } k \leq n, \\
(ii) & \quad it = 0, \text{ otherwise}.
\end{align*}
\]

Proof: Define $P[\text{natural number}] = f(k + 1) - 1)$. Set $n_3 = n -' k + 1$. Consider $p$ being a finite sequence such that $\text{len}p = n_3$ and for every $i$ such that $i \in \text{dom}p$ holds $p(i) = P(i)$ from \[6, \text{Sch. 2}]. \text{rng}_p \subseteq \mathbb{C}$. $p(1 + 0) = f(k + 0)$ and ... and $p(1 + (n -' k)) = f(k + (n -' k))$ by \[4, (11)]]. □

(10) If $(f, k) + \ldots + (f, n) \neq 0$, then there exists $i$ such that $k \leq i \leq n$ and $i \in \text{dom}f$.

Proof: Consider $h$ such that $(f, k) + \ldots + (f, n) = \sum h$ and

\[
\begin{align*}
(i) & \quad h(0 + 1) = f(0 + k) \text{ and } \ldots \text{ and } h(n -' k + 1) = f(n -' k + k). \\
(ii) & \quad it = 0, \text{ otherwise}.
\end{align*}
\]

Now we state the propositions:

(9) Suppose $k \leq n$. Then there exists $h$ such that

\[
\begin{align*}
(i) & \quad (f, k) + \ldots + (f, n) = \sum h, \text{ and} \\
(ii) & \quad \text{len} h = n -' k + 1, \text{ and} \\
(iii) & \quad h(0 + 1) = f(0 + k) \text{ and } \ldots \text{ and } h(n -' k + 1) = f(n -' k + k).
\end{align*}
\]

Proof: Define $P[\text{natural number}] = f(k + 1) - 1)$. Set $n_3 = n -' k + 1$. Consider $p$ being a finite sequence such that $\text{len}p = n_3$ and for every $i$ such that $i \in \text{dom}p$ holds $p(i) = P(i)$ from \[6, \text{Sch. 2}]. \text{rng}_p \subseteq \mathbb{C}$. $p(1 + 0) = f(k + 0)$ and ... and $p(1 + (n -' k)) = f(k + (n -' k))$ by \[4, (11)]]. □

(11) $(f, k) + \ldots + (f, k) = f(k)$. The theorem is a consequence of (9).

(12) If $k \leq n + 1$, then $(f, k) + \ldots + (f, (n + 1)) = ((f, k) + \ldots + (f, n)) + f(n + 1)$. The theorem is a consequence of (11) and (9).

(13) If $k \leq n$, then $(f, k) + \ldots + (f, n) = f(k) + ((f, (k + 1)) + \ldots + (f, n))$. The theorem is a consequence of (11) and (9).

(14) If $k \leq m \leq n$, then $((f, k) + \ldots + (f, m)) + ((f, (m + 1)) + \ldots + (f, n)) = (f, k) + \ldots + (f, n)$.

Proof: Define $P[\text{natural number}] = ((f, k) + \ldots + (f, m)) + ((f, (m + 1)) + \ldots + (f, (m + 1)))). \text{P}[0]$ by \[4, (13)]]. If $P[i]$, then $P[i + 1]$ by \[4, (11)]], (12). $P[i]$ from \[4, \text{Sch. 2}]. □
(15) If \( k > \text{len } h \), then \((h, k) + \ldots + (h, n) = 0\). The theorem is a consequence of (9).

(16) If \( n \geq \text{len } h \), then \((h, k) + \ldots + (h, n) = (h, k) + \ldots + (h, \text{len } h)\). The theorem is a consequence of (15) and (12).

(17) \((h, 0) + \ldots + (h, k) = (h, 1) + \ldots + (h, k)\). The theorem is a consequence of (13).

(18) \((h, 1) + \ldots + (h, \text{len } h) = \sum h\). The theorem is a consequence of (9).

(19) \((g \sim h, k) + \ldots + (g \sim h, n) = ((g, k) + \ldots + (g, n)) + ((h, (k - \text{len } g)) + \ldots + (h, (n - \text{len } g))). The theorem is a consequence of (11), (15), (16), (17), and (14).

Let us consider \( n \) and \( k \). Let \( f \) be a real-valued finite sequence. One can check that \((f, k) + \ldots + (f, n)\) is real.

Let \( f \) be a natural-valued finite sequence. Note that \((f, k) + \ldots + (f, n)\) is natural.

Let \( f \) be a complex-valued function. Assume \( \text{dom } f \cap \mathbb{N} \) is finite. The functor \((f, n) + \ldots \) yielding a complex number is defined by

(Def. 2) for every \( k \) such that for every \( i \) such that \( i \in \text{dom } f \) holds \( i \leq k \) holds

\[ it = (f, n) + \ldots + (f, k). \]

Let us consider \( h \). One can check that the functor \((h, n) + \ldots \) yields a complex number and is defined by the term

(Def. 3) \((h, n) + \ldots + (h, \text{len } h)\).

Let \( n \) be a natural number and \( h \) be a natural-valued finite sequence. Let us note that \((h, n) + \ldots \) is natural.

Now we state the propositions:

(20) Let us consider a finite, complex-valued function \( f \). Then \( f(n) + (f, (n + 1)) + \ldots = (f, n) + \ldots \). The theorem is a consequence of (13).

(21) \( \sum h = (h, 1) + \ldots \)

(22) \( \sum h = h(1) + (h, 2) + \ldots \). The theorem is a consequence of (18) and (20).

The scheme \( TT \) deals with complex-valued finite sequences \( f, g \) and natural numbers \( a, b \) and non zero natural numbers \( n, k \) and states that

(Sch. 1) \((f, a) + \ldots = (g, b) + \ldots \)

provided

- for every \( j \), \((f, (a + j \cdot n)) + \ldots + (f, (a + j \cdot n + (n - 1))) = (g, (b + j \cdot k)) + \ldots + (g, (b + j \cdot k + (k - 1))).\)
3. Power Function

Let $r$ be a real number and $f$ be a real-valued function. The functor $rf$ yielding a real-valued function is defined by

(Def. 4) \ dom it = \ dom f \ and \ for \ every \ x \ such \ that \ x \in \ dom f \ holds \ it(x) = rf(x).

Let $n$ be a natural number and $f$ be a natural-valued function. One can verify that $nf$ is natural-valued.

Let $r$ be a real number and $f$ be a real-valued finite sequence. One can check that $rf$ is finite sequence-like and $rf$ is $(\text{len } f)$-element.

Let $f$ be a one-to-one, natural-valued function. Observe that $(2 + n)f$ is one-to-one.

(23) Let us consider real numbers $r, s$. Then $r^{(s)} = (r^s)$.

(24) Let us consider a real number $r$, and real-valued finite sequences $f, g$. Then $rf \cdot g = rf \bowtie rg$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every increasing, natural-valued finite sequence $f$ such that $n > 1$ and $f(\text{len } f) < s_1$ and $f \neq \emptyset$ holds $\sum f < 2 \cdot n^{(\text{len } f)}$. For every natural-valued finite sequence $f$ such that $n > 1$ and $\text{len } f = 1$ holds $\sum f < 2 \cdot n^{(\text{len } f)}$ by [26] (25), [19] (83), [6] (40), [11] (73)]. $\mathcal{P}[0]$ by [26] (25), [4] (25)]. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by [4] (8), (25), (13)], [26] (25)]. $\mathcal{P}[i]$ from [4] Sch. 2]. $\sum f = n^f(1) + (n^f, 2) + \ldots$. □

(25) Let us consider a real-valued function $f$, and a function $g$. Then $2f \cdot g = 2f \cdot g$. Proof: Define $h = 2f$. Set $f_5 = f \cdot g$. For every $i$ such that $1 \leq i \leq \text{len } f_5$ holds $f_5(i) = (r_2 \bowtie r_3)(i)$ by [26] (25), [6] (25)]. □

(26) Let us consider an increasing, natural-valued finite sequence $f$. If $n > 1$, then $n^f(1) + (n^f, 2) + \ldots < 2 \cdot n^{(\text{len } f)}$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every increasing, natural-valued finite sequence $f$ such that $n > 1$ and $f(\text{len } f) < s_1$ and $f \neq \emptyset$ holds $\sum f < 2 \cdot n^{(\text{len } f)}$ by [26] (25), [19] (83), [6] (40), [11] (73)]. $\mathcal{P}[0]$ by [26] (25), [4] (25)]. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by [4] (8), (25), (13)], [26] (25)]. $\mathcal{P}[i]$ from [4] Sch. 2]. $\sum f = n^f(1) + (n^f, 2) + \ldots$. □

(27) Let us consider increasing, natural-valued finite sequences $f_1, f_2$. Suppose $n > 1$ and $n^{f_1}(1) + (n^{f_1}, 2) + \ldots = n^{f_2}(1) + (n^{f_2}, 2) + \ldots$. Then $f_1 = f_2$.

Proof: For every natural-valued finite sequence $f$ such that $n > 1$ and $\sum f < 0$ holds $f = \emptyset$ by [11] (85), [19] (83)]. Define $\mathcal{P}[\text{natural number}] \equiv$ for every increasing, natural-valued finite sequences $f_1, f_2$ such that $n > 1$ and $\sum f_1 < s_1$ and $\sum f_1 = \sum f_2$ holds $f_1 = f_2$. $\mathcal{P}[0]$. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by (21), (22), [4] (8)], [11] (72)]. $\mathcal{P}[i]$ from [4] Sch. 2]. $n^{f_1}(1) + (n^{f_1}, 2) + \ldots = \sum f_1$. $n^{f_2}(1) + (n^{f_2}, 2) + \ldots = \sum f_2$. □

(28) Let us consider a natural-valued function $f$. If $n > 1$, then $\text{Coim}(nf, nk) = \text{Coim}(f, k)$. Proof: $\text{Coim}(nf, nk) \subseteq \text{Coim}(f, k)$ by [17] (30)]. □
(29) Let us consider natural-valued functions $f_1$, $f_2$. Suppose $n > 1$. Then $f_1$ and $f_2$ are fiberwise equipotent if and only if $n^{f_1}$ and $n^{f_2}$ are fiberwise equipotent. Proof: If $f_1$ and $f_2$ are fiberwise equipotent, then $n^{f_1}$ and $n^{f_2}$ are fiberwise equipotent by $[9]$ (72)], $[17]$ (30)], (28). For every object $x$, $\text{Coim}(f_1, x) = \text{Coim}(f_2, x)$ by $[9]$ (72)], $[17]$ (30)], (28). □

(30) Let us consider one-to-one, natural-valued finite sequences $f_1$, $f_2$. Suppose $n > 1$ and $n^{f_1}(1) + (n^{f_1}, 2) + \ldots = n^{f_2}(1) + (n^{f_2}, 2) + \ldots$. Then $\text{rng} f_1 = \text{rng} f_2$.

Proof: Reconsider $F_1 = f_1$, $F_2 = f_2$ as a finite sequence of elements of $\mathbb{R}$. Set $s_1 = \text{sort}_a F_1$. Set $s_2 = \text{sort}_a F_2$. $n^{F_1}$ and $n^{s_1}$ are fiberwise equipotent. $n^{F_2}$ and $n^{s_2}$ are fiberwise equipotent. For every extended reals $e_1$, $e_2$ such that $e_1$, $e_2 \in \text{dom} s_1$ and $e_1 < e_2$ holds $s_1(e_1) < s_1(e_2)$ by $[16]$ (2)], $[2]$ (77)]. For every extended reals $e_1$, $e_2$ such that $e_1$, $e_2 \in \text{dom} s_2$ and $e_1 < e_2$ holds $s_2(e_1) < s_2(e_2)$ by $[16]$ (2)], $[2]$ (77)]. $\sum n^{s_1} = n^{s_1}(1) + (n^{s_1}, 2) + \ldots$. $\sum n^{f_1} = n^{f_1}(1) + (n^{f_1}, 2) + \ldots$. $\sum n^{s_1} = \sum n^{s_2}$. $n^{s_1}(1) + (n^{s_1}, 2) + \ldots = n^{s_2}(1) + (n^{s_2}, 2) + \ldots$ and $s_1$ is increasing and natural-valued. □

(31) There exists an increasing, natural-valued finite sequence $f$ such that $n = 2^f(1) + (2^f, 2) + \ldots$.

Proof: Set $D = \text{digits}(n, 2)$. Consider $d$ being a finite 0-sequence of $\mathbb{N}$ such that $\text{dom} d = \text{dom} D$ and for every natural number $i$ such that $i \in \text{dom} d$ holds $d(i) = D(i) \cdot 2^i$ and $\text{value}(D, 2) = \sum d$. Define $P[\text{natural number}] \equiv$ if $\$1 \leq \text{len} d$, then there exists an increasing, natural-valued finite sequence $f$ such that $(\text{len} f = 0$ or $f(\text{len} f) < \$1$) and $\sum 2^f = \sum (d|\$1$). $P[0 \text{ qua natural number}]$ by $[11]$ (72)]. If $P[i]$, then $P[i + 1]$ by $[4]$ (13)], $[29]$ (86)], $[20]$ (65)], $[4]$ (25), (23)]. $P[i]$ from $[4]$ Sch. 2]. Consider $f$ being an increasing, natural-valued finite sequence such that $\text{len} f = 0$ or $f(\text{len} f) < \text{len} d$ and $\sum 2^f = \sum (d|\text{len} d)$. $\sum 2^f = 2^f(1) + (2^f, 2) + \ldots$. □

4. Value-based Function (Re)Organization

Let $o$ be a function yielding function and $x$, $y$ be objects. The functor $o_{x,y}$ yielding a set is defined by the term

(Def. 5) $o(x)(y)$.

Let $F$ be a function yielding function. We say that $F$ is double one-to-one if and only if

(Def. 6) for every objects $x_1$, $x_2$, $y_1$, $y_2$ such that $x_1 \in \text{dom} F$ and $y_1 \in \text{dom}(F(x_1))$ and $x_2 \in \text{dom} F$ and $y_2 \in \text{dom}(F(x_2))$ and $F_{x_1,y_1} = F_{x_2,y_2}$ holds $x_1 = x_2$ and $y_1 = y_2$. 
Let $D$ be a set. Observe that every finite sequence of elements of $D^*$ which is empty is also double one-to-one and there exists a function yielding function which is double one-to-one and there exists a finite sequence of elements of $D^*$ which is double one-to-one.

Let $F$ be a double one-to-one, function yielding function and $x$ be an object. One can check that $F(x)$ is one-to-one.

Let $F$ be a one-to-one function. One can check that $\langle F \rangle$ is double one-to-one.

Now we state the propositions:

(32) Let us consider a function yielding function $f$. Then $f$ is double one-to-one if and only if for every $x$, $f(x)$ is one-to-one and for every $x$ and $y$ such that $x \neq y$ holds $\text{rng}(f(x))$ misses $\text{rng}(f(y))$.

(33) Let us consider a set $D$, and double one-to-one finite sequences $f_1$, $f_2$ of elements of $D^*$. Suppose $\text{Values } f_1$ misses $\text{Values } f_2$. Then $f_1 \smallsetminus f_2$ is double one-to-one. The theorem is a consequence of (1).

Let $D$ be a finite set.

A double reorganization of $D$ is a double one-to-one finite sequence of elements of $D^*$ and is defined by

(Def. 7) Values $it = D$.

Now we state the propositions:

(i) $\emptyset$ is a double reorganization of $\emptyset$, and
(ii) $\langle \emptyset \rangle$ is a double reorganization of $\emptyset$.

(35) Let us consider a finite set $D$, and a one-to-one, onto finite sequence $F$ of elements of $D$. Then $\langle F \rangle$ is a double reorganization of $D$.

(36) Let us consider finite sets $D_1$, $D_2$. Suppose $D_1$ misses $D_2$. Let us consider a double reorganization $o_1$ of $D_1$, and a double reorganization $o_2$ of $D_2$. Then $o_1 \smallsetminus o_2$ is a double reorganization of $D_1 \cup D_2$. The theorem is a consequence of (33) and (2).

(37) Let us consider a finite set $D$, a double reorganization $o$ of $D$, and a one-to-one finite sequence $F$. Suppose $i \in \text{dom } o$ and $\text{rng } F \cap D \subseteq \text{rng}(o(i))$. Then $o \smallsetminus (i, F)$ is a double reorganization of $\text{rng } F \cup (D \ \text{rng}(o(i)))$.

**Proof:** Set $r_1 = \text{rng } F$. Set $o_3 = o(i)$. Set $r_4 = \text{rng } o_3$. Set $o_4 = o \smallsetminus (i, F)$. $\text{rng } o_4 \subseteq (r_1 \cup (D \ \text{rng}(o(i))))$ by [7] (31), (32)]. $o_4$ is double one-to-one by [7] (32), (1). Values $o_4 \subseteq r_1 \cup (D \ \text{rng}(o(i)))$ by (1), [7] (31), (32)]. $D \ \text{rng}(o(i)) \subseteq r_4 \subseteq \text{Values } o_4$ by (1), [7] (32)]. $r_1 \subseteq \text{Values } o_4$. □

Let $D$ be a finite set and $n$ be a non zero natural number. One can check that there exists a double reorganization of $D$ which is $n$-element.

Let $D$ be a finite, natural-membered set, $o$ be a double reorganization of $D$, and $x$ be an object. One can verify that $o(x)$ is natural-valued.
Now we state the propositions:

(38) Let us consider a non empty finite sequence $F$, and a finite function $G$. Suppose $\text{rng } G \subseteq \text{rng } F$. Then there exists a $(\text{len } F)$-element double reorganization $o$ of $\text{dom } G$ such that for every $n$, $F(n) = G(o_{n,1})$ and ... and $F(n) = G(o_{n,\text{len}(o(n))})$.

**Proof:** Set $D = \text{dom } G$. Set $d = \text{the one-to-one, onto finite sequence of elements of } D$. Define $P[\text{natural number}] \equiv \text{if } \exists k \forall n, F(n) = G(o_{n,1})$ and ... and $F(n) = G(o_{n,\text{len}(o(n))})$. $P[0]$. If $P[i]$, then $P[i + 1]$ by [4, (13)], [9, (11), (12)]. $P[i]$ from [4 Sch. 2]. □

(39) Let us consider a non empty finite sequence $F$, and a finite sequence $G$. Suppose $\text{rng } G \subseteq \text{rng } F$. Then there exists a $(\text{len } F)$-element double reorganization $o$ of $\text{dom } G$ such that for every $n$, $o(n)$ is increasing and $F(n) = G(o_{n,1})$ and ... and $F(n) = G(o_{n,\text{len}(o(n))})$.

**Proof:** Define $P[\text{natural number}] \equiv \text{if } \exists k \forall n, F(n) = G(o_{n,1})$ and ... and $F(n) = G(o_{n,\text{len}(o(n))})$. $P[0]$. If $P[i]$, then $P[i + 1]$ by [4, (13)], [9, (11), (12)]. $P[i]$ from [4 Sch. 2]. □

Let $f$ be a finite function, $o$ be a double reorganization of $\text{dom } f$, and $x$ be an object. One can check that $f \cdot o(x)$ is finite sequence-like and there exists a finite sequence which is complex-functions-valued and finite sequence-yielding.

Let $f$ be a function yielding function and $g$ be a function. We introduce $g \circ f$ as a synonym of $[g,f]$.

One can check that $g \circ f$ is function yielding.
Let $f$ be a $(\text{dom } g)^*\text{-valued finite sequence}$. One can check that $g \circ f$ is finite sequence-yielding.

Let $x$ be an object. Let us note that $(g \circ f)(x)$ is $(\text{len } (f(x)))\text{-element}.

Let $f$ be a function yielding finite sequence. One can verify that $g \circ f$ is finite sequence-like and $g \circ f$ is $(\text{len } f)\text{-element}.

Let $f$ be a function yielding function and $g$ be a complex-valued function. One can check that $g \circ f$ is complex-functions-valued.

Let $g$ be a natural-valued function. One can check that $g \circ f$ is natural-functions-valued.

Let us consider a function yielding function $f$ and a function $g$. Now we state the propositions:

(40) Values $g \circ f = g^\circ(\text{Values } f)$.

**Proof:** Set $g_3 = g \circ f$. Values $g_3 \subseteq g^\circ(\text{Values } f)$ by (1), [9, (11), (12)]. Consider $b$ being an object such that $b \in \text{dom } g$ and $b \in \text{Values } f$ and
g(b) = a. Consider x, y being objects such that x ∈ dom f and y ∈ dom(f(x)) and b = f(x)(y). □

(41) \((g \circ f)(x) = g \cdot f(x)\).

Now we state the proposition:

(42) Let us consider a function yielding function f, a finite sequence g, and objects x, y. Then \((g \circ f)_{x,y} = g(f_{x,y})\). The theorem is a consequence of (41).

Let f be a complex-functions-valued, finite sequence-yielding function. The functor \(\sum f\) yielding a complex-valued function is defined by

\[
\text{dom } it = \text{dom } f \quad \text{and for every set } x, \quad it(x) = \sum (f(x)).
\]

Let f be a complex-functions-valued, finite sequence-yielding finite sequence. One can verify that \(\sum f\) is finite sequence-like and \(\sum f\) is (len f)-element.

Let f be a natural-functions-valued, finite sequence-yielding function. One can verify that \(\sum f\) is natural-valued.

Let f, g be complex-functions-valued finite sequences. One can check that \(f \circ g\) is complex-functions-valued.

Let f, g be extended real-valued finite sequences. One can verify that \(f \circ g\) is extended real-valued.

Let f be a complex-functions-valued function and X be a set. One can check that \(f\upharpoonright X\) is complex-functions-valued.

Let f be a finite sequence-yielding function. One can check that \(f\upharpoonright X\) is finite sequence-yielding.

Let F be a complex-valued function. One can check that \(\langle F \rangle\) is complex-functions-valued.

Let us consider finite sequences f, g. Now we state the propositions:

(43) If \(f \circ g\) is finite sequence-yielding, then f is finite sequence-yielding and g is finite sequence-yielding.

(44) If \(f \circ g\) is complex-functions-valued, then f is complex-functions-valued and g is complex-functions-valued.

Now we state the propositions:

(45) Let us consider a complex-valued finite sequence f. Then \(\sum \langle f \rangle = \langle \sum f \rangle\).

(46) Let us consider complex-functions-valued, finite sequence-yielding finite sequences f, g. Then \(\sum (f \circ g) = \sum f \circ \sum g\).

PROOF: For every i such that \(1 \leq i \leq \text{len } f + \text{len } g\) holds \((\sum (f \circ g))(i) = (\sum f \circ \sum g)(i)\) by [26, (25)], [6, (25)]. □

(47) Let us consider a complex-valued finite sequence f, and a double reorganization o of dom f. Then \(\sum f = \sum \sum (f \circ o)\).
Proof: Define $P$[natural number] $\equiv$ for every complex-valued finite sequence $f$ for every double reorganization $o$ of dom $f$ such that $\text{len } f = \$_1$ holds $\sum f = \sum \sum (f \circ o)$. $P[0]$ by [26] (29), [11] (72), [23] (11), [11] (81). If $P[i]$, then $P[i + 1]$ by [4] (11), [26] (25), (1), [12] (116). $P[i]$ from [4, Sch. 2]. ☐

Let us note that $\mathbb{N}^*$ is natural-functions-membered and $\mathbb{C}^*$ is complex-functions-membered.

Now we state the proposition:

(48) Let us consider a finite sequence $f$ of elements of $\mathbb{C}^*$.

Then $\sum (\text{the concatenation of } C \circ f) = \sum \sum f$.

Proof: Set $C =$ the concatenation of $\mathbb{C}$. Define $P$[natural number] $\equiv$ for every finite sequence $f$ of elements of $\mathbb{C}^*$ such that $\text{len } f = \$_1$ holds $\sum (C \circ f) = \sum \sum f$. $P[0]$. If $P[i]$, then $P[i + 1]$ by [8] (19), (16), (46), (45). $P[i]$ from [4, Sch. 2]. ☐

Let $f$ be a finite function.

A valued reorganization of $f$ is a double reorganization of dom $f$ and is defined by (Def. 9) for every $n$, there exists $x$ such that $x = f(it_{n,1})$ and ... and $x = f(it_{n,\text{len}(it(n))})$ and for every natural numbers $n_1$, $n_2$, $i_1$, $i_2$ such that $i_1 \in \text{dom}(it(n_1))$ and $i_2 \in \text{dom}(it(n_2))$ and $f(it_{n_1,i_1}) = f(it_{n_2,i_2})$ holds $n_1 = n_2$.

Now we state the propositions:

(49) Let us consider a finite function $f$, and a valued reorganization $o$ of $f$.

Then

(i) $\text{rng}((f \circ o)(n)) = \emptyset$, or

(ii) $\text{rng}((f \circ o)(n)) = \{f(o_{n,1})\}$ and $1 \in \text{dom}(o(n))$.

Proof: Consider $y$ such that $y \in \text{rng}((f \circ o)(n))$. Consider $x$ such that $x \in \text{dom}((f \circ o)(n))$ and $(f \circ o)(n)(x) = y$. $n \in \text{dom}(f \circ o)$. Consider $w$ being an object such that $w = f(o_{n,1})$ and ... and $w = f(o_{n,\text{len}(o(n))}).$ $\text{rng}((f \circ o)(n)) \subseteq \{f(o_{n,1})\}$ by [9] (11), (12], [26] (25). ☐

(50) Let us consider a finite sequence $f$, and valued reorganizations $o_1$, $o_2$ of $f$. Suppose $\text{rng}((f \circ o_1)(i)) = \text{rng}((f \circ o_2)(i))$. Then $\text{rng}(o_1(i)) = \text{rng}(o_2(i))$.

(51) Let us consider a finite sequence $f$, a complex-valued finite sequence $g$, and double reorganizations $o_1$, $o_2$ of dom $g$. Suppose $o_1$ is a valued reorganization of $f$ and $o_2$ is a valued reorganization of $f$ and $\text{rng}((f \circ o_1)(i)) = \text{rng}((f \circ o_2)(i))$. Then $(\sum (g \circ o_1))(i) = (\sum (g \circ o_2))(i)$. The theorem is a consequence of (41).
References

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Euler’s Partition Theorem

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Summary. In this article we prove the Euler’s Partition Theorem which states that the number of integer partitions with odd parts equals the number of partitions with distinct parts. The formalization follows H.S. Wilf’s lecture notes [28] (see also [1]).

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The notation and terminology used in this paper have been introduced in the following articles: [22], [2], [3], [17], [7], [16], [19], [14], [15], [23], [9], [10], [24], [5], [18], [6], [11], [29], [12], [26], and [13].

1. Preliminaries

From now on $x$, $y$ denote objects and $i$, $j$, $k$, $m$, $n$ denote natural numbers.

Let $r$ be an extended real number. One can verify that $\langle r \rangle$ is extended real-valued and $\langle r \rangle$ is decreasing, increasing, non-decreasing, and non-increasing.

Now we state the proposition:

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Let us consider non-decreasing, extended real-valued finite sequences \( f, g \). If \( f(\text{len} f) \leq g(1) \), then \( f \triangleleft g \) is non-decreasing.

**Proof:** Set \( f_3 = f \triangleleft g \). For every extended reals \( e_1, e_2 \) such that \( e_1, e_2 \in \text{dom} f_3 \) and \( e_1 \leq e_2 \) holds \( f_3(e_1) \leq f_3(e_2) \) by [7, (25)], [25, (25)]. □

Let \( R \) be a binary relation. We say that \( R \) is odd-valued if and only if

(Def. 1) \( \text{rng} R \subseteq \mathbb{N}_{\text{odd}} \).

(2) \( n \in \mathbb{N}_{\text{odd}} \) if and only if \( n \) is odd.

Let us note that every binary relation which is odd-valued is also non-zero and natural-valued.

Let \( F \) be a function. Observe that \( F \) is odd-valued if and only if the condition (Def. 2) is satisfied.

(Def. 2) for every \( x \) such that \( x \in \text{dom} F \) holds \( F(x) \) is an odd natural number.

One can check that every binary relation which is empty is also odd-valued.

Let \( i \) be an odd natural number. Let us observe that \( (i) \) is odd-valued.

Let \( f, g \) be odd-valued finite sequences. Note that \( f \triangleleft g \) is odd-valued and every binary relation which is \( \mathbb{N}_{\text{odd}} \)-valued is also odd-valued.

Let \( n \) be a natural number. A partition of \( n \) is a non-zero, non-decreasing, natural-valued finite sequence and is defined by

(Def. 3) \( \sum it = n \).

Now we state the proposition:

(3) \( \emptyset \) is a partition of 0.

Let \( n \) be a natural number. Observe that there exists a partition of \( n \) which is odd-valued and there exists a partition of \( n \) which is one-to-one.

Let us observe that sethood property holds for partitions of \( n \).

Let \( f \) be an odd-valued finite sequence.

An odd organization of \( f \) is a valued reorganization of \( f \) and is defined by

(Def. 4) \( 2 \cdot n - 1 = f(it_{n,1}) \) and ... and \( 2 \cdot n - 1 = f(it_{n,\text{len}(it(n))}) \).

(4) Let us consider an odd-valued finite sequence \( f \), and a double reorganization \( o \) of \( \text{dom} f \). Suppose for every \( n, 2 \cdot n - 1 = f(o_{n,1}) \) and ... and \( 2 \cdot n - 1 = f(o_{n,\text{len}(o(n))}) \). Then \( o \) is an odd organization of \( f \).

**Proof:** For every \( n \), there exists \( x \) such that \( x = f(o_{n,1}) \) and ... and \( x = f(o_{n,\text{len}(o(n))}) \). For every natural numbers \( n_1, n_2, i_1, i_2 \) such that \( i_1 \in \text{dom}(o(n_1)) \) and \( i_2 \in \text{dom}(o(n_2)) \) and \( f(o_{n_1,i_1}) = f(o_{n_2,i_2}) \) holds \( n_1 = n_2 \) by [25, (25)]. □

(5) Let us consider an odd-valued finite sequence \( f \), a complex-valued finite sequence \( g \), and double reorganizations \( o_1, o_2 \) of \( \text{dom} g \). Suppose \( o_1 \) is an odd organization of \( f \) and \( o_2 \) is an odd organization of \( f \). Then \( (\sum (g \circ o_1))(i) = (\sum (g \circ o_2))(i) \).
Proof: For every double reorganizations \( o_1, o_2 \) of \( \text{dom} \, g \) such that \( o_1 \) is an odd organization of \( f \) and \( o_2 \) is an odd organization of \( f \) holds \( \text{rng}((f \circ o_1)(n)) \subseteq \text{rng}((f \circ o_2)(n)) \) by [19] (49), (1), [25] (29), (25). □

(6) Let us consider a partition \( p \) of \( n \). Then there exists an odd-valued finite sequence \( O \) and there exists a natural-valued finite sequence \( a \) such that \( \text{len} \, O = \text{len} \, p = \text{len} \, a \) and \( p = O \cdot 2^a \) and \( p(1) = O(1) \cdot 2^{a(1)} \) and ... and \( p(\text{len} \, p) = O(\text{len} \, p) \cdot 2^{a(\text{len} \, p)} \).

Proof: Define \( \mathcal{P}[\text{object}, \text{object}] \equiv \) for every \( i \) and \( j \) such that \( p(\$1) = 2^i \cdot (2 \cdot j + 1) \) holds \( \$2 = \langle 2 \cdot j + 1, i \rangle \). For every \( k \) such that \( k \in \text{Seg} \, \text{len} \, p \) there exists \( x \) such that \( \mathcal{P}[k, x] \) by [20] (1), [4] (4). Consider \( O_3 \) being a finite sequence such that \( \text{dom} \, O_3 = \text{Seg} \, \text{len} \, p \) and for every \( k \) such that \( k \in \text{Seg} \, \text{len} \, p \) holds \( \mathcal{P}[k, O_3(k)] \) from [7] Sch. 1. Define \( Q(\text{object}) = O_3(\$1)_2 \).

Consider \( O \) being a finite sequence such that \( \text{len} \, O = \text{len} \, p \) and for every \( k \) such that \( k \in \text{dom} \, O \) holds \( O(k) = Q(k) \) from [7] Sch. 2. For every \( x \) such that \( x \in \text{dom} \, O \) holds \( O(x) \) is an odd natural number by [20] (1). Define \( T(\text{object}) = O_3(\$1)_2 \). Consider \( A \) being a finite sequence such that \( \text{len} \, A = \text{len} \, p \) and for every \( k \) such that \( k \in \text{dom} \, A \) holds \( A(k) = T(k) \) from [7] Sch. 2. For every \( x \) such that \( x \in \text{dom} \, A \) holds \( A(x) \) is natural by [20] (1). Set \( O_2 = O \cdot 2^A \), \( p(1) = O(1) \cdot 2^{A(1)} \) and ... and \( p(\text{len} \, p) = O(\text{len} \, p) \cdot 2^{A(\text{len} \, p)} \) by [25] (25), [20] (1). For every \( i \) such that \( i \in \text{dom} \, p \) holds \( p(i) = O_2(i) \) by [25] (25). □

(7) Let us consider a finite set \( D \), and a function \( f \) from \( D \) into \( \mathbb{N} \). Then there exists a finite sequence \( K \) of elements of \( D \) such that for every element \( d \) of \( D \), \( \text{Coim}(K, d) = f(d) \).

Proof: Define \( \mathcal{P}[\text{natural number}] \equiv \) for every finite set \( D \) such that \( \text{len} \, D = \$1 \) for every function \( f \) from \( D \) into \( \mathbb{N} \), there exists a finite sequence \( K \) of elements of \( D \) such that for every element \( d \) of \( D \), \( \text{Coim}(K, d) = f(d) \).


(8) Let us consider complex-valued finite sequences \( f_1, f_2, g_1, g_2 \). Suppose \( \text{len} \, f_1 = \text{len} \, g_1 \). Then \( (f_1 \sim f_2) \cdot (g_1 \sim g_2) = (f_1 \cdot g_1) \sim (f_2 \cdot g_2) \).

(9) Let us consider natural-valued finite sequences \( f, K \). Suppose for every \( i, \text{Coim}(K, i) = f(i) \). Then \( \sum K = 1 \cdot f(1) + 2 \cdot f(2) + (\text{id}_{\text{dom} \, f} \cdot f, 3) + \ldots \).

Proof: Define \( \mathcal{P}[\text{natural number}] \equiv \) for every natural-valued finite sequences \( f, K \) such that \( \text{len} \, f = \$1 \) and for every \( i, \text{Coim}(K, i) = f(i) \) holds \( \sum K = (\text{id}_{\text{dom} \, f} \cdot f, 1) + \ldots \mathcal{P}[0] \) by [25] (25), [9] (72), [19] (20), (22). If \( \mathcal{P}[i] \), then \( \mathcal{P}[i + 1] \) by [25] (55), [5] (13), [7] (59), [8] (51). \( \mathcal{P}[i] \) from [5] Sch. 2. □

(10) Let us consider a natural-valued finite sequence \( g \), and a double reorgani-
zation $s_1$ of $\text{dom} \ g$. Then there exists a $(2 \cdot \text{len}(s_1))$-element finite sequence $K$ of elements of $\mathbb{N}$ such that for every $j$, $K(2 \cdot j) = 0$ and $K(2 \cdot j - 1) = g(s_{1,j,1}) + ((g \circ s_1)(j), 2) + \ldots$. Proof: Define $\mathcal{P}[$object, object$] \equiv$ if $\text{rng} f \subseteq \mathbb{N}$ by \[ \text{rng} f = \text{dom} \ f \text{ (25)} \text{ by } \[25\ (25)], \[5\ (13)], \[19\ (15)]. \square$

2. Euler Transformation

Now we state the proposition:

(11) Let us consider a one-to-one partition $d$ of $n$. Then there exists an odd-valued partition $e$ of $n$ such that for every natural number $j$ for every odd-valued finite sequence $O_1$ for every natural-valued finite sequence $a_1$ such that $\text{len} O_1 = \text{len} d = \text{len} a_1$ and $d = O_1 \cdot 2^{a_1}$ for every double reorganization $s_1$ of $d$ such that $1 = O_1(s_{1,1,1})$ and ... and $1 = O_1(s_{1,\text{len}(s_1)(1)})$ and $3 = O_1(s_{1,2,1})$ and ... and $3 = O_1(s_{1,\text{len}(s_1)(2)})$ and $5 = O_1(s_{1,3,1})$ and ... and $5 = O_1(s_{1,\text{len}(s_1)(3)})$ and for every $i$, $2 \cdot i - 1 = O_1(s_{1,1,i})$ and $2 \cdot i - 1 = O_1(s_{1,\text{len}(s_1)(i)})$ holds $\text{Coim}(e, 1) = 2^{a_1}(s_{1,1,1}) + ((2^{a_1} \circ s_1)(1), 2) + \ldots$ and $\text{Coim}(e, 3) = 2^{a_1}(s_{1,2,1}) + ((2^{a_1} \circ s_1)(2), 2) + \ldots$ and $\text{Coim}(e, 5) = 2^{a_1}(s_{1,3,1}) + ((2^{a_1} \circ s_1)(3), 2) + \ldots$ and $\text{Coim}(e, j \cdot 2 - 1) = 2^{a_1}(s_{1,1,j}) + ((2^{a_1} \circ s_1)(j), 2) + \ldots$.

Proof: Consider $O$ being an odd-valued finite sequence, $a$ being a natural-valued finite sequence such that $\text{len} O = \text{len} d = \text{len} a$ and $d = O \cdot 2^a$ and $d(1) = O(1) \cdot 2^{a(1)}$ and ... and $d(\text{len} d) = O(\text{len} d) \cdot 2^{\text{len}(\text{len} d)}$. $n = d(1) + ((d, 2) + \ldots + (d, \text{len} d))$ by $[19\ (22)]$. $n = 2^{a(1)} \cdot O(1) + 2^{a(2)} \cdot O(2) + (O \cdot 2^a, 3) + \ldots + (O \cdot 2^a, \text{len} d)$ by $[19\ (20)], [25\ (25)]$. Reconsider $s_1$ of $\text{dom} \ g$ as a double reorganization of $d$ and $2^a$.

Consider $\mu$ being a $(2 \cdot \text{len}(s_1))$-element finite sequence of elements of $\mathbb{N}$ such that for every $j$, $\mu(2 \cdot j) = 0$ and $\mu(2 \cdot j - 1) = 2^a(s_{1,j,1}) + ((2^a \circ s_1)(j), 2) + \ldots$. Set $\alpha = a \cdot s_1(1)$. Set $\beta = a \cdot s_1(2)$. Set $\gamma = a \cdot s_1(3)$. $n = (2^a(1) + (2^a, 2) + \ldots) \cdot 1 + (2^a(1) + (2^a, 2) + \ldots) \cdot 3 + (2^a(1) + (2^a, 2) + \ldots) \cdot 5 + (\text{id}_{\text{dom} \mu} \cdot \mu, 7) + \ldots$ by $[25\ (29)], [19\ (41)], [25\ (25)], [19\ (12)]$. $n = \mu(1) \cdot 1 + \mu(3) \cdot 3 + \mu(5) \cdot 5 + (\text{id}_{\text{dom} \mu} \cdot \mu, 7) + \ldots$ by $[19\ (42), (41), (25)]$. Consider $K$ being an odd-valued finite sequence such that $K$ is non-decreasing and for every $i$, $\text{Coim}(K, i) = \mu(i)$. $n = \text{Coim}(K, 1) \cdot 1 + \text{Coim}(K, 3) \cdot 3 + \text{Coim}(K, 5) \cdot 5 + (\text{id}_{\text{dom} \mu} \cdot \mu, 7) + \ldots$ $n = \sum K$ by $[19\ (20)], (9)$. For every $j$ such
that $1 \leq j \leq \text{len } d$ holds $O(j) = O_1(j)$ and $a(j) = a_1(j)$ by \cite{25} (25), \cite{22} (9), \cite{4} (4). For every $j$, $\text{Coim}(K,j \cdot 2 - 1) = 2^{a_1(\text{sort}1,j_1)} + ((2^{a_1} \circ \text{sort}1)(j), 2) + \ldots$ by \cite{19} (42), \cite{23} (29), \cite{9} (72), \cite{19} (22). □

Let $n$ be a natural number and $p$ be a one-to-one partition of $n$. The Euler transformation $p$ yielding an odd-valued partition of $n$ is defined by

\begin{align*}
\text{Def. 5} \quad & \text{for every odd-valued finite sequence } O \text{ and for every natural-valued finite sequence } a \text{ such that } \text{len } O = \text{len } p = \text{len } a \text{ and } p = O \cdot 2^a \text{ for every double reorganization } s_1 \text{ of dom } p \text{ such that } 1 = O(s_{11,1}) \text{ and } \ldots \text{ and } 1 = O(s_{11,\text{len}(s_{11,1}))} \text{ and } 3 = O(s_{12,1}) \text{ and } \ldots \text{ and } 3 = O(s_{12,\text{len}(s_{12,1}))} \text{ and } 5 = O(s_{13,1}) \text{ and } \ldots \text{ and } 5 = O(s_{13,\text{len}(s_{13,1}))} \text{ and for every } i, 2 \cdot i - 1 = O(s_{1i,1}) \text{ and } \ldots \text{ and } 2 \cdot i - 1 = O(s_{1i,\text{len}(s_{1i,1}))}) \text{ holds } \text{Coim}(it, 1) = 2^{a_1(s_{11,1})} + ((2^a \circ s_1)(1), 2) + \ldots \text{ and } \text{Coim}(it, 3) = 2^a(s_{12,1}) + ((2^a \circ s_1)(2), 2) + \ldots \text{ and } \text{Coim}(it, 5) = 2^a(s_{13,1}) + ((2^a \circ s_1)(3), 2) + \ldots \text{ and } \text{Coim}(it, j \cdot 2 - 1) = 2^a(s_{1j,1}) + ((2^a \circ s_1)(j), 2) + \ldots .
\end{align*}

Now we state the proposition:

(12) Let us consider a natural number $n$, a one-to-one partition $p$ of $n$, and an odd-valued partition $e$ of $n$. Then $e$ is the Euler transformation $p$ if and only if for every odd-valued finite sequence $O$ and for every natural-valued finite sequence $a$ and for every odd organization $s_1$ of $O$ such that $\text{len } O = \text{len } p = \text{len } a$ and $p = O \cdot 2^a$ for every $j$, $\text{Coim}(e, j \cdot 2 - 1) = ((2^a \circ s_1)(j), 1) + \ldots$.

Proof: If $e$ is the Euler transformation $p$, then for every odd-valued finite sequence $O$ and for every natural-valued finite sequence $a$ and for every odd organization $s_1$ of $O$ such that $\text{len } O = \text{len } p = \text{len } a$ and $p = O \cdot 2^a$ for every $j$, $\text{Coim}(e, j \cdot 2 - 1) = ((2^a \circ s_1)(j), 1) + \ldots$ by \cite{25} (29), \cite{19} (42), (20). For every $j$ and for every odd-valued finite sequence $O$ and for every natural-valued finite sequence $a$ such that $\text{len } O = \text{len } p = \text{len } a$ and $p = O \cdot 2^a$ for every double reorganization $s_1$ of dom $p$ such that $1 = O(s_{11,1})$ and $\ldots$ and $1 = O(s_{11,\text{len}(s_{11,1}))}$ and $3 = O(s_{12,1})$ and $\ldots$ and $3 = O(s_{12,\text{len}(s_{12,1}))}$ and $5 = O(s_{13,1})$ and $\ldots$ and $5 = O(s_{13,\text{len}(s_{13,1}))}$ and for every $i$, $2 \cdot i - 1 = O(s_{1i,1})$ and $\ldots$ and $2 \cdot i - 1 = O(s_{1i,\text{len}(s_{1i,1}))})$ holds $\text{Coim}(e, 1) = 2^a(s_{11,1}) + ((2^a \circ s_1)(1), 2) + \ldots$ and $\text{Coim}(e, 3) = 2^a(s_{12,1}) + ((2^a \circ s_1)(2), 2) + \ldots$ and $\text{Coim}(e, 5) = 2^a(s_{13,1}) + ((2^a \circ s_1)(3), 2) + \ldots$ and $\text{Coim}(e, j \cdot 2 - 1) = 2^a(s_{1j,1}) + ((2^a \circ s_1)(j), 2) + \ldots$ by \cite{25} (29), (4), \cite{19} (42), (20). □

One can verify that every real-valued function which is one-to-one and non-decreasing is also increasing.
(13) Let us consider an odd-valued finite sequence $O$, a natural-valued finite sequence $a$, and an odd organization $s$ of $O$. Suppose $\text{len } O = \text{len } a$ and $O \cdot 2^a$ is one-to-one. Then $(a \odot s)(i)$ is one-to-one.

**Proof:** $(a \odot s)(i)$ is one-to-one by [9, (11), (12)], [25, (25)]. □

(14) Let us consider one-to-one partitions $p_1, p_2$ of $n$. Suppose the Euler transformation $p_1 = \text{the Euler transformation } p_2$. Then $p_1 = p_2$.

(15) Let us consider an odd-valued partition $e$ of $n$. Then there exists a one-to-one partition $p$ of $n$ such that $e = \text{the Euler transformation } p$.

**Proof:** Define $K(\text{object}) = \text{Coim}(e, S_1)$. Consider $H$ being a finite sequence such that $\text{len } H = n$ and for every $k$ such that $k \in \text{dom } H$ holds $H(k) = K(k)$ from [7, Sch. 2]. $\text{rng } H \subseteq \mathbb{N}$. $e = \sum(\text{idseq}(n) \cdot H)$ by [25, (25)], [5, (14)], [9, (72)]. Define $F[\text{natural number}, \text{object}] \equiv \text{there exists an increasing, natural-valued finite sequence } f \text{ such that } H(S_1) = 2^f(1) + 2^f(2) + \ldots \text{ and } S_2 = S_1 \cdot 2^f$. There exists a finite sequence $p$ of elements of $\mathbb{N}^*$ such that $\text{dom } p = \text{Seg } \text{len } H$ and for every $k$ such that $k \in \text{Seg } \text{len } H$ holds $F[k, p(k)]$ by [19, (31)]. Consider $p$ being a finite sequence of elements of $\mathbb{N}^*$ such that $\text{dom } p = \text{Seg } \text{len } H$ and for every $k$ such that $k \in \text{Seg } \text{len } H$ holds $F[k, p(k)]$. For every $k$ such that $p(k) \neq \emptyset$ holds $k$ is odd by [19, (83)], [12, (85)], [19, (22)], [9, (72)]. Set $N = \text{the concatenation of } \mathbb{N}$. Set $n_3 = N \circ p$. Set $s_2 = \text{sort}_a n_3$. $s_2$ is a one-to-one partition of $n$ by [19, (1)], [25, (25)], [12, (45)], [18, (83)]. For every odd-valued finite sequence $O$ and for every natural-valued finite sequence $a$ and for every odd organization $s_1$ of $O$ such that $\text{len } O = \text{len } s_2 = \text{len } a$ and $s_2 = O \cdot 2^a$ for every $j$, $\text{Coim}(e, j \cdot 2^1) = ((2^a \odot s_1)(j), 1) + \ldots$ by [25, (29)], [5, (14)], [9, (72)], [25, (25)]. □

3. Main Theorem

Now we state the proposition:

(16) **Euler’s partition theorem:**

the set of all $p$ where $p$ is an odd-valued partition of $n$ =
the set of all $p$ where $p$ is a one-to-one partition of $n$. The theorem is a consequence of (15) and (14).

**References**

Euler’s Partition Theorem

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Introduction to Diophantine Approximation

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Summary. In this article we formalize some results of Diophantine approximation, i.e. the approximation of an irrational number by rationals. A typical example is finding an integer solution \((x, y)\) of the inequality \(|x\theta - y| \leq \frac{1}{x}\), where \(\theta\) is a real number. First, we formalize some lemmas about continued fractions. Then we prove that the inequality has infinitely many solutions by continued fractions. Finally, we formalize Dirichlet’s proof (1842) of existence of the solution \([12], [1]\).

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The notation and terminology used in this paper have been introduced in the following articles: [24], [2], [6], [22], [14], [5], [11], [7], [8], [28], [20], [26], [3], [25], [19], [4], [9], [32], [15], [13], [21], [30], [31], [18], [23], [29], and [10].

1. Irrational Numbers and Continued Fractions

From now on \(i, j, k, m, n, m_1, n_1\) denote natural numbers, \(a, r, r_1, r_2\) denote real numbers, \(m_0, c_3, c_1\) denote integers, and \(x_1, x_2, o\) denote objects.

Now we state the proposition:

\((1) \quad (i) \quad r = (\text{rfs } r)(0), \text{ and} \)

\((ii) \quad r = (\text{scf } r)(0) + \frac{1}{(\text{rfs } r)(1)}, \text{ and} \)

\((iii) \quad (\text{rfs } r)(n) = (\text{scf } r)(n) + \frac{1}{(\text{rfs } r)(n+1)}). \)

Let us assume that \(r\) is irrational. Now we state the propositions:
(2) $(rfs \ r)(n)$ is irrational.

**Proof:** Reconsider $r_3 = (rfs \ r)(n)$ as a real number. $(scf \ r_3)(m) = (scf \ r)(n + m)$ and $(rfs \ r_3)(m) = (rfs \ r)(n + m)$. Consider $n_1$ such that for every $m_1$ such that $m_1 \geq n_1$ holds $(scf \ r_3)(m_1) = 0$. For every $m_1$ such that $m_1 \geq n_1$ holds $(scf \ r)(n + m_1) = 0$. For every $m$ such that $m \geq n_1 + n$ holds $(scf \ r)(m) = 0$ by [28 (3)]. □

(3) (i) $(rfs \ r)(n) \neq 0$, and
(ii) $(rfs \ r)(1) \cdot (rfs \ r)(2) \neq 0$, and
(iii) $(scf \ r)(1) \cdot (rfs \ r)(2) + 1 \neq 0$.

**Proof:** $(rfs \ r)(n) \neq 0$ by [21 (28), (42)]. $(rfs \ r)(1) \neq 0$ and $(rfs \ r)(2) \neq 0$. $(rfs \ r)(1) = (scf \ r)(1) + (1/(rfs \ r)(1 + 1))$. □

(4) (i) $(scf \ r)(n) < (rfs \ r)(n) < (scf \ r)(n) + 1$, and
(ii) $1 < (rfs \ r)(n + 1)$.

The theorem is a consequence of (2) and (1).

(5) $0 < (scf \ r)(n + 1)$. The theorem is a consequence of (4).

Let us consider $r$ and $n$. Observe that $(cn \ r)(n)$ is integer.

Let us assume that $r$ is irrational. Now we state the propositions:

(6) $(cd \ r)(n + 1) \geq (cd \ r)(n)$.

**Proof:** Define $P[nat \ number] \equiv (cd \ r)(n) \leq (cd \ r)(n + 1)$. $P[0]$ by (4), [28 (7)]. For every natural number $n$ such that $P[n]$ holds $P[n + 1]$ by (4), [28 (7)], [21 (51)]. For every natural number $n$, $P[n]$ from [3 Sch. 2]. □

(7) $(cd \ r)(n) \geq 1$.

**Proof:** Define $P[nat \ number] \equiv (cd \ r)(n) \geq 1$. For every natural number $n$ such that $P[n]$ holds $P[n + 1]$. For every natural number $n$, $P[n]$ from [3 Sch. 2]. □

(8) $(cd \ r)(n + 2) > (cd \ r)(n + 1)$. The theorem is a consequence of (5) and (7).

(9) $(cd \ r)(n) \geq n$.

**Proof:** Define $P[nat \ number] \equiv (cd \ r)(n) \geq n$. For every natural number $n$ such that $P[n]$ holds $P[n + 1]$ by (7), (5), [21 (40)]. For every natural number $n$, $P[n]$ from [3 Sch. 2]. □

Now we state the proposition:

(10) If $c_3 = (cn \ r)(n)$ and $c_1 = (cd \ r)(n)$ and $c_3 \neq 0$, then $c_3$ and $c_1$ are relatively prime.

Let us assume that $r$ is irrational. Now we state the propositions:

(11) (i) $(cd \ r)(n + 1) \cdot (rfs \ r)(n + 2) + (cd \ r)(n) > 0$, and
\[ (cdr)(n+1) \cdot (rfsr)(n+2) - (cdr)(n) > 0. \]
The theorem is a consequence of (7), (4), and (6).

(12) \[ (cdr)(n+1) \cdot ((cdr)(n+1) \cdot (rfsr)(n+2) + (cdr)(n)) > 0. \]
The theorem is a consequence of (7) and (11).

(13) \[ r = (cnr)(n+1) \cdot (rfsr)(n+2) + (cnr)(n)/((cdr)(n+1) \cdot (rfsr)(n+2) + (cdr)(n)). \]

**Proof:** Define \( \mathcal{P}[\text{natural number}] \equiv r = (cnr)(n_1+1) \cdot (rfsr)(n_1+2) + (cnr)(n_1)/((cdr)(n_1+1) \cdot (rfsr)(n_1+2) + (cdr)(n_1)). \) \( \mathcal{P}[0]. \) For every natural number \( n \) such that \( \mathcal{P}[n] \) holds \( \mathcal{P}[n+1]. \) For every natural number \( n, \mathcal{P}[n] \) from \( [1, \text{Sch. } 2]. \) □

(14) \[ \left( \frac{cnr(n+1)}{cdr(n+1)} \right) - r = \frac{(-1)^n}{(cdr)(n+1) \cdot (cdr)(n+1) \cdot (rfsr)(n+2) + (cdr)(n)}. \]
The theorem is a consequence of (7), (11), and (13).

Now we state the propositions:

(15) If \( r \) is irrational and \( n \) is even and \( n > 0, \) then \( r > \frac{(cnr)(n)}{(cdr)(n)}. \)

The theorem is a consequence of (12) and (14).

(16) If \( r \) is irrational and \( n \) is odd, then \( r < \frac{(cnr)(n)}{(cdr)(n)}. \)

The theorem is a consequence of (12) and (14).

(17) Suppose \( r \) is irrational and \( n > 0. \) Then \( |r - \frac{(cnr)(n)}{(cdr)(n)}| + |r - \frac{(cnr)(n+1)}{(cdr)(n+1)}| = \left| \frac{cnr(n)}{(cdr)(n)} - \frac{cnr(n+1)}{(cdr)(n+1)} \right|. \)

The theorem is a consequence of (15) and (16).

Let us assume that \( r \) is irrational. Now we state the propositions:

(18) \[ |r - \frac{(cnr)(n)}{(cdr)(n)}| > 0. \]

(19) \[ (cdr)(n+2) > 2 \cdot (cdr)(n). \]

The theorem is a consequence of (5), (7), and (6).

(20) \[ |r - \frac{(cnr)(n+1)}{(cdr)(n+1)}| < \frac{1}{(cdr)(n+1) \cdot (cdr)(n+2)}. \]

The theorem is a consequence of (7), (4), and (14).

(21) \n
(i) \[ |r \cdot (cdr)(n+1) - (cnr)(n+1)| < |r \cdot (cdr)(n) - (cnr)(n)|, \]

(ii) \[ |r - \frac{(cnr)(n+1)}{(cdr)(n+1)}| < |r - \frac{(cnr)(n)}{(cdr)(n)}|. \]

The theorem is a consequence of (13), (11), (4), (7), (18), and (6).

Now we state the propositions:

(22) If \( r \) is irrational and \( m > n, \) then \( |r - \frac{(cnr)(n)}{(cdr)(n)}| > |r - \frac{(cnr)(m)}{(cdr)(m)}|. \)

**Proof:** Define \( \mathcal{P}[\text{natural number}] \equiv |r - \frac{(cnr)(n)}{(cdr)(n)}| > |r - \frac{(cnr)(n+1)}{(cdr)(n+1)}|. \) \( \mathcal{P}[0]. \) For every natural number \( k \) such that \( \mathcal{P}[k] \) holds \( \mathcal{P}[k+1]. \) For every natural number \( k, \mathcal{P}[k] \) from \( [1, \text{Sch. } 2]. \) □

(23) If \( r \) is irrational, then \[ |r - \frac{(cnr)(n)}{(cdr)(n)}| < \frac{1}{(cdr)(n)^2}. \]
\textbf{Proof:} \(|r - ((cnr)(n)/(cdr)(n))| < 1/(cdr)(n)^2\) by (28) \((43)\), (7), \([16, (1)]\), (6). □

(24) Let us consider a subset \(S\) of \(\mathbb{Q}\), and \(r\). Suppose \(r\) is irrational and 
\(S = \{p, \text{where } p \text{ is an element of } \mathbb{Q} : |r - p| < 1/(\text{den } p)^2\}\). Then \(S\) is infinite.

\textbf{Proof:} Define \(\mathcal{F}(\text{natural number}) = (cnr)(S_1 + 1)/(cdr)(S_1 + 1)\). Consider 
\(f\) being a sequence of real numbers such that for every natural number \(n\), 
\(f(n) = \mathcal{F}(n)\) from \([17\text{ Sch. 1}]\). For every real number \(o\) such that 
\(o \in \text{rng } f\) holds \(o \in S\) by \([21\text{ (50)}], (7), [15\text{ (28)}], [16\text{ (1)}]\). \(f\) is one-to-one. □

(25) If \(r\) is irrational, then \(\text{cocf } r\) is convergent and \(\lim \text{cocf } r = r\).

\textbf{Proof:} For every real number \(p\) such that \(0 < p\) there exists \(n\) such that 
for every \(m\) such that \(n \leq m\) holds \(|(\text{cocf } r)(m) - r| < p\) by \([27\text{ (25)}], [28\text{ (3)}], [17\text{ (8)}], [6\text{ (52)}]\) □

2. Integer Solution of \(|x\theta - y| \leq 1/x\)

Let us observe that there exists a natural number which is greater than 1. From now on \(t\) denotes a greater than 1 natural number.

Let us consider \(t\). The functor \(\text{EDI}(t)\) yielding a sequence of subsets of \(\mathbb{R}\) is defined by

(Def. 1) for every natural number \(n\), \(it(n) = [n/t, n + 1/t]\).

Now we state the propositions:

(26) (The partial unions of \(\text{EDI}(t))\)(\(i\)) = \([0, i + 1/t]\].
\textbf{Proof:} Define \(\mathcal{P}[\text{natural number}] \equiv \text{(the partial unions of } \text{EDI}(t))(S_1) = \[0, S_1 + 1/i].\) For every natural number \(k\) such that \(\mathcal{P}[k]\) holds \(\mathcal{P}[k + 1]\). For every natural number \(n\), \(\mathcal{P}[n]\) from \([3\text{ Sch. 2}]. □

(27) Let us consider a real number \(r\), and a natural number \(i\). If \([r \cdot t] = i\), 
then \(r \in \text{(EDI}(t))\)(\(i\)).

(28) If \(r_1, r_2 \in \text{(EDI}(t))\)(\(i\)), then \(|r_1 - r_2| < t^{-1}\).

(29) (The partial unions of \(\text{EDI}(t))\)(\(t - 1\)) = \([0, 1]\]. The theorem is a consequence of (26).

(30) Let us consider a real number \(r\). Suppose \(r \in [0, 1]\). Then there exists a 
natural number \(i\) such that 
\begin{enumerate}
  \item \(i \leq t - 1\), and
  \item \(r \in \text{(EDI}(t))\)(\(i\)).
\end{enumerate}
The theorem is a consequence of (29).

(31) Let us consider a real number \(r\), and a natural number \(i\). If \(r \in \text{(EDI}(t))\)(\(i\)), 
then \([r \cdot t] = i\).
(32) Let us consider a real number \( r \). Suppose \( r \in [0,1] \). Then there exists a natural number \( i \) such that

(i) \( i \leq t - 1 \), and

(ii) \( \lfloor r \cdot t \rfloor = i \).

The theorem is a consequence of (30) and (31).

Let us consider \( t \) and \( a \). The functor \( \text{FDP}(t,a) \) yielding a finite sequence of elements of \( \mathbb{Z}_t \) is defined by

(Def. 2) \( \text{len } it = t + 1 \) and for every \( i \) such that \( i \in \text{dom } it \) holds \( it(i) = \lfloor \frac{(i - 1) \cdot a}{t} \rfloor \).

Let us note that \( \text{rng FDP}(t,a) \) is non empty.

Now we state the proposition:

(33) \( \text{rng FDP}(t,a) \subset \text{dom FDP}(t,a) \).

Let us consider \( t \) and \( a \). One can verify that \( \text{FDP}(t,a) \) is non one-to-one.

3. Proof of Dirichlet’s Theorem

Now we state the proposition:

(34) **Dirichlet’s Approximation Theorem:**

There exist integers \( x, y \) such that

(i) \( \vert x \cdot a - y \vert < 1/t \), and

(ii) \( 0 < x < t \).

The theorem is a consequence of (27) and (28).

**References**


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Finite Product of Semiring of Sets

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Summary. We formalize that the image of a semiring of sets \[17\] by an injective function is a semiring of sets. We offer a non-trivial example of a semiring of sets in a topological space \[21\]. Finally, we show that the finite product of a semiring of sets is also a semiring of sets \[21\] and that the finite product of a classical semiring of sets \[8\] is a classical semiring of sets. In this case, we use here the notation from the book of Aliprantis and Border \[1\].

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The notation and terminology used in this paper have been introduced in the following articles: \[9\], \[2\], \[3\], \[4\], \[22\], \[7\], \[15\], \[23\], \[10\], \[11\], \[6\], \[12\], \[20\], \[26\], \[27\], \[19\], \[14\], \[16\], \[25\], \[18\], and \[13\].

1. Preliminaries

From now on on \(X_1, X_2, X_3, X_4\) denote sets.

Now we state the propositions:

(1) (i) \(X_1 \cap X_4 \setminus (X_2 \cup X_3)\) misses \(X_1 \setminus ((X_2 \cup X_3) \cup X_4)\), and
   (ii) \(X_1 \cap X_4 \setminus (X_2 \cup X_3)\) misses \((X_1 \cap X_3) \cap X_4 \setminus X_2\), and
   (iii) \(X_1 \setminus ((X_2 \cup X_3) \cup X_4)\) misses \((X_1 \cap X_3) \cap X_4 \setminus X_2\).

(2) \((X_1 \setminus X_2) \setminus (X_3 \setminus X_4) = (X_1 \setminus (X_2 \cup X_3)) \cup (X_1 \cap X_4 \setminus X_2)\).

(3) \((X_1 \setminus (X_2 \cup X_3)) \cup (X_1 \cap X_4 \setminus X_2) = ((X_1 \cap X_4 \setminus (X_2 \cup X_3)) \cup (X_1 \setminus ((X_2 \cup X_3) \cup X_4))) \cup ((X_1 \cap X_3) \cap X_4 \setminus X_2)\).
(4) \((X_1 \setminus X_2) \setminus (X_3 \setminus X_4) = ((X_1 \cap X_4 \setminus (X_2 \cup X_3)) \cup (X_1 \setminus ((X_2 \cup X_3) \cup X_4))) \cup ((X_1 \cap X_3) \cap X_4 \setminus X_2)\). The theorem is a consequence of (2) and (3).

(5) \(\bigcup \{X_1, X_2, X_3\} = (X_1 \cup X_2) \cup X_3\).

2. The Direct Image of a Semiring of Sets by an Injective Function

Now we state the proposition:

(6) Let us consider sets \(T, S\), a function \(f\) from \(T\) into \(S\), and a family \(G\) of subsets of \(T\). Then \(f^\circ G = \{f^\circ A, \text{ where } A \in G\}\).

Let \(T, S\) be sets, \(f\) be a function from \(T\) into \(S\), and \(G\) be a finite family of subsets of \(T\). Let us note that \(f^\circ G\) is finite.

Let \(f\) be a function and \(A\) be a countable set. Let us note that \(f^\circ A\) is countable.

The scheme \textit{FraenkelCountable} deals with a set \(\mathcal{A}\) and a set \(\mathcal{X}\) and a unary functor \(\mathcal{F}\) yielding a set and states that

\[(\text{Sch. 1}) \quad \{\mathcal{F}(w), \text{ where } w \text{ is an element of } \mathcal{A} : w \in \mathcal{X}\} \text{ is countable provided}
\]

- \(\mathcal{X}\) is countable.

Let \(T, S\) be sets, \(f\) be a function from \(T\) into \(S\), and \(G\) be a countable family of subsets of \(T\). Let us note that \(f^\circ G\) is countable.

Let \(X, Y\) be sets, \(S\) be a family of subsets of \(X\) with the empty element, and \(f\) be a function from \(X\) into \(Y\). One can verify that \(f^\circ S\) has the empty element.

Now we state the propositions:

(7) Let us consider sets \(X, Y\), a function \(f\) from \(X\) into \(Y\), and families \(S_1, S_2\) of subsets of \(X\). If \(S_1 \subseteq S_2\), then \(f^\circ S_1 \subseteq f^\circ S_2\). The theorem is a consequence of (6).

(8) Let us consider sets \(X, Y\), a \(\cap\)-closed family \(S\) of subsets of \(X\), and a function \(f\) from \(X\) into \(Y\). Suppose \(f\) is one-to-one. Then \(f^\circ S\) is a \(\cap\)-closed family of subsets of \(Y\).

(9) Let us consider non empty sets \(X, Y\), a \(\cap_{fp}\)-closed family \(S\) of subsets of \(X\), and a function \(f\) from \(X\) into \(Y\). Suppose \(f\) is one-to-one. Then \(f^\circ S\) is a \(\cap_{fp}\)-closed family of subsets of \(Y\).

(10) Let us consider non empty sets \(X, Y\), a \(\setminus_{fp}\)-closed family \(S\) of subsets of \(X\), and a function \(f\) from \(X\) into \(Y\). Suppose \(f\) is one-to-one and \(f^\circ S\) is not empty. Then \(f^\circ S\) is a \(\setminus_{fp}\)-closed family of subsets of \(Y\).
Proof: Reconsider \( f_1 = f^\circ S \) as a family of subsets of \( Y \). \( f_1 \) is \( \subseteq_{\text{fp}} \)-closed by \([10]\ (64), (87)], \([11]\ (103)], \[26]\ (123)]. \( \Box \)

(11) Let us consider non empty sets \( X, Y \), a \( \setminus_{\text{fp}} \)-closed family \( S \) of subsets of \( X \), and a function \( f \) from \( X \) into \( Y \). Suppose \( f \) is one-to-one. Then \( f^\circ S \) is a \( \setminus_{\text{fp}} \)-closed family of subsets of \( Y \).

(12) Let us consider non empty sets \( X, Y \), a semiring \( S \) of sets of \( X \), and a function \( f \) from \( X \) into \( Y \). If \( f \) is one-to-one, then \( f^\circ S \) is a semiring of sets of \( Y \).

3. The Set of Set Differences of All Elements of a Semiring of Sets

Now we state the proposition:

(13) Let us consider a 1-element finite sequence \( X \). Suppose \( X(1) \) is not empty. Then there exists a function \( I \) from \( X(1) \) into \( \prod X \) such that

(i) \( I \) is one-to-one and onto, and

(ii) for every object \( x \) such that \( x \in X(1) \) holds \( I(x) = \langle x \rangle \).

Let \( X \) be a set. Observe that \( 2^X \) is \( \cap \)-closed and there exists a \( \cap \)-closed family of subsets of \( X \) which has the empty element and there exists a \( \cap \)-closed family of subsets of \( X \) with the empty element which is \( \cup \)-closed.

Let \( X, Y \) be non empty sets. Let us observe that \( X \setminus Y \) is non empty.

Now we state the proposition:

(14) Let us consider a set \( X \), and a family \( S \) of subsets of \( X \) with the empty element. Then \( S \setminus S \) is the set of all \( A \setminus B \) where \( A, B \) are elements of \( S \).

Let \( X \) be a set and \( S \) be a family of subsets of \( X \) with the empty element.

The functor \( \text{semidiff} \) \( S \) yielding a family of subsets of \( X \) is defined by the term (Def. 1) \( S \setminus S \).

Now we state the proposition:

(15) Let us consider a set \( X \), a family \( S \) of subsets of \( X \) with the empty element, and an object \( x \). Suppose \( x \in \text{semidiff} \) \( S \). Then there exist elements \( A, B \) of \( S \) such that \( x = A \setminus B \). The theorem is a consequence of (14).

Let \( X \) be a set and \( S \) be a family of subsets of \( X \) with the empty element. Observe that \( \text{semidiff} \) \( S \) has the empty element.

Let \( S \) be a \( \cap \)-closed, \( \cup \)-closed family of subsets of \( X \) with the empty element.

Note that \( \text{semidiff} \) \( S \) is \( \cap \)-closed and \( \setminus_{\text{fp}} \)-closed.

Now we state the proposition:

(16) Let us consider a set \( X \), and a \( \cap \)-closed, \( \cup \)-closed family \( S \) of subsets of \( X \) with the empty element. Then \( \text{semidiff} \) \( S \) is a semiring of sets of \( X \).
4. The Collection of All Locally Closed Sets $LC(X, \tau)$ of a Topological Space $(X, \tau)$

Let $T$ be a non empty topological space. The functor $LC(T)$ yielding a family of subsets of $\Omega_T$ is defined by the term

(Def. 2) \{ $A \cap B$, where $A, B$ are subsets of $T : A$ is open and $B$ is closed$\}$.

Let us note that $LC(T)$ is $\cap$-closed and $\setminus fp$-closed and has the empty element.

(17) Let us consider a non empty topological space $T$. Then $LC(T)$ is a semiring of sets of $\Omega_T$.

5. The Finite Product of Semirings of Sets

Let $n$ be a natural number. Note that there exists an $n$-element finite sequence which is non-empty.

Let $n$ be a non zero natural number and $X$ be a non-empty, $n$-element finite sequence.

A semiring family of $X$ is an $n$-element finite sequence and is defined by

(Def. 3) for every natural number $i$ such that $i \in \text{Seg} n$ holds $it(i)$ is a semiring of sets of $X(i)$.

In the sequel $n$ denotes a non zero natural number and $X$ denotes a non-empty, $n$-element finite sequence. Now we state the propositions:

(18) Let us consider a semiring family $S$ of $X$. Then $\text{dom } S = \text{dom } X$.

(19) Let us consider a semiring family $S$ of $X$, and a natural number $i$. If $i \in \text{Seg} n$, then $\bigcup(S(i)) \subseteq X(i)$.

(20) Let us consider a function $f$, and an $n$-element finite sequence $X$. If $f \in \prod X$, then $f$ is an $n$-element finite sequence.

Let $n$ be a non zero natural number and $X$ be an $n$-element finite sequence. The functor $\text{SemiringProduct } X$ yielding a set is defined by

(Def. 4) for every object $f$, $f \in it$ iff there exists a function $g$ such that $f = \prod g$ and $g \in \prod X$.

Now we state the propositions:

(21) Let us consider an $n$-element finite sequence $X$.

Then $\text{SemiringProduct } X \subseteq 2^{\bigcup \text{dom } X}$.

(22) Let us consider a semiring family $S$ of $X$. Then $\text{SemiringProduct } S$ is a family of subsets of $\prod X$.

Proof: Reconsider $S_1 = \text{SemiringProduct } S$ as a subset of $2^{\bigcup \text{dom } S}$. $S_1 \subseteq 2^{\prod X}$ by $[3, (9)]$, (18), $[7, (89)]$, (19). □
Let us consider a non-empty, 1-element finite sequence $X$. Then $\prod X = \{\langle x \rangle \mid x \in X(1)\}$. The theorem is a consequence of (13).

One can check that $\prod(\emptyset)$ is empty. Now we state the propositions:

Let us consider a non-empty set $x$. Then $\prod(\langle x \rangle) = \{\langle y \rangle \mid y \in x\}$. The theorem is a consequence of (23).

Let us consider a non-empty, 1-element finite sequence $X$, and a semiring family $S$ of $X$. Then $\text{SemiringProduct}(S) = \{\langle s \rangle \mid s \in S(1)\}$. PROOF: $S$ is non-empty by (18), [7, (3)]. $\prod S = \{\langle s \rangle \mid s \in S(1)\}$.

Let us consider sets $x, y$. Now we state the propositions:

$\prod(\langle x \rangle) \cap \prod(\langle y \rangle) = \prod(\langle x \cap y \rangle)$. The theorem is a consequence of (24).

$\prod(\langle x \rangle) \setminus \prod(\langle y \rangle) = \prod(\langle x \setminus y \rangle)$. The theorem is a consequence of (24).

Let us consider a non-empty, 1-element finite sequence $X$ and a semiring family $S$ of $X$. Now we state the propositions:

the set of all $\prod(\langle s \rangle)$ where $s$ is an element of $S(1)$ is a semiring of sets of the set of all $\langle x \rangle$ where $x$ is an element of $X(1)$. The theorem is a consequence of (24), (26), and (27).

$\text{SemiringProduct}(S)$ is a semiring of sets of $\prod X$. The theorem is a consequence of (23), (25), and (28).

Let us consider sets $X_1, X_2$, a semiring $S_1$ of sets of $X_1$, and a semiring $S_2$ of sets of $X_2$. Then the set of all $s_1 \times s_2$ where $s_1$ is an element of $S_1$, $s_2$ is an element of $S_2$ is a semiring of sets of $X_1 \times X_2$.

Let us consider a non-empty, $n$-element finite sequence $X_3$, a non-empty, 1-element finite sequence $X_1$, a semiring family $S_3$ of $X_3$, and a semiring family $S_1$ of $X_1$. Suppose $\text{SemiringProduct}(S_3)$ is a semiring of sets of $\prod X_3$ and $\text{SemiringProduct}(S_1)$ is a semiring of sets of $\prod X_1$. Let us consider a family $S_4$ of subsets of $\prod X_3 \times \prod X_1$. Suppose $S_4 = \{\langle s_1 \times s_2 \rangle \mid s_1 \in S_3, s_2 \in S_1\}$. Then there exists a function $I$ from $\prod X_3 \times \prod X_1$ into $\prod(\langle X_3 \setminus X_1 \rangle)$ such that

(i) $I$ is one-to-one and onto, and

(ii) for every finite sequences $x, y$ such that $x \in \prod X_3$ and $y \in \prod X_1$ holds $I(x, y) = x \setminus y$, and

(iii) $I(S_4) = \text{SemiringProduct}(S_3 \setminus S_1)$.

PROOF: $\bigcup(S_1(1)) \subseteq X_1(1)$. Consider $I$ being a function from $\prod X_3 \times \prod X_1$ into $\prod(\langle X_3 \setminus X_1 \rangle)$ such that $I$ is one-to-one and $I$ is onto and for every finite
sequences $x, y$ such that $x \in \prod X_3$ and $y \in \prod X_1$ holds $I(x, y) = x \bowtie y$. 
$I \circ S_4 = \text{SemiringProduct}(S_3 \bowtie S_1)$ by (25), (20), [7 (89)], [24 (153)]. □

(32) Let us consider a non-empty, $n$-element finite sequence $X_3$, a non-empty, 1-element finite sequence $X_1$, a semiring family $S_3$ of $X_3$, and a semiring family $S_1$ of $X_1$. Suppose $\text{SemiringProduct} S_3$ is a semiring of sets of $\prod X_3$ and $\text{SemiringProduct} S_1$ is a semiring of sets of $\prod X_1$. Then $\text{SemiringProduct}(S_3 \bowtie S_1)$ is a semiring of sets of $\prod(X_3 \bowtie X_1)$. The theorem is a consequence of (30), (31), (9), and (10).

(33) Let us consider a semiring family $S$ of $X$. Then $\text{SemiringProduct} S$ is a semiring of sets of $\prod X$. PROOF: Define $P[\text{non zero natural number}] \equiv$ for every non-empty, $\text{\#}_1$-element finite sequence $X$ for every semiring family $S$ of $X$, $\text{SemiringProduct} S$ is a semiring of sets of $\prod X$. $P[1]$. For every non zero natural number $n$, $P[n]$ from [5, Sch. 10]. □

Let $n$ be a non zero natural number, $X$ be a non-empty, $n$-element finite sequence, and $S$ be a semiring family of $X$. We say that $S$ is $\cap$-closed yielding if and only if

(Def. 5) for every natural number $i$ such that $i \in \text{Seg} n$ holds $S(i)$ is $\cap$-closed.

Note that there exists a semiring family of $X$ which is $\cap$-closed yielding.

6. THE FINITE PRODUCT OF CLASSICAL SEMIRINGS OF SETS

Let $X$ be a set. Note that there exists a semiring of sets of $X$ which is $\cap$-closed.

Let us consider a non-empty, 1-element finite sequence $X$ and a $\cap$-closed yielding semiring family $S$ of $X$. Now we state the propositions:

(34) the set of all $\prod \langle s \rangle$ where $s$ is an element of $S(1)$ is a $\cap$-closed semiring of sets of the set of all $\langle x \rangle$ where $x$ is an element of $X(1)$. The theorem is a consequence of (26) and (28).

(35) $\text{SemiringProduct} S$ is a $\cap$-closed semiring of sets of $\prod X$. The theorem is a consequence of (23), (25), and (34).

Now we state the propositions:

(36) Let us consider sets $X_1$, $X_2$, a $\cap$-closed semiring $S_1$ of sets of $X_1$, and a $\cap$-closed semiring $S_2$ of sets of $X_2$. Then the set of all $s_1 \times s_2$ where $s_1$ is an element of $S_1$, $s_2$ is an element of $S_2$ is a $\cap$-closed semiring of sets of $X_1 \times X_2$.

(37) Let us consider a non-empty, $n$-element finite sequence $X_3$, a non-empty, 1-element finite sequence $X_1$, a $\cap$-closed yielding semiring family $S_3$ of $X_3$, and a $\cap$-closed yielding semiring family $S_1$ of $X_1$. Suppose $\text{SemiringProduct} S_3$ is a semiring of sets of $\prod X_3$ and $\text{SemiringProduct} S_1$ is a semiring of sets of $\prod X_1$. Then $\text{SemiringProduct}(S_3 \bowtie S_1)$ is a semiring of sets of $\prod(X_3 \bowtie X_1)$. The theorem is a consequence of (30), (31), (9), and (10).
S₃ is a \( \cap \)-closed semiring of sets of \( \prod X₃ \) and SemiringProduct \( S₁ \) is a \( \cap \)-closed semiring of sets of \( \prod X₁ \). Then SemiringProduct \( (S₃ \cap S₁) \) is a \( \cap \)-closed semiring of sets of \( \prod (X₃ \cap X₁) \). The theorem is a consequence of (30), (31), (36), (8), and (10).

Let us consider \( n \) and \( X \). Let \( S \) be a \( \cap \)-closed yielding semiring family of \( X \). One can check that SemiringProduct \( S \) is \( \cap \)-closed.

(38) Let us consider a \( \cap \)-closed yielding semiring family \( S \) of \( X \).
Then SemiringProduct \( S \) is a \( \cap \)-closed semiring of sets of \( \prod X \).

7. MEASURABLE RECTANGLE

Let \( n \) be a non zero natural number and \( X \) be a non-empty, \( n \)-element finite sequence.
A classical semiring family of \( X \) is an \( n \)-element finite sequence and is defined by

(Def. 6) for every natural number \( i \) such that \( i \in \text{Seg} n \) holds \( \text{it}(i) \) is a semi-diff-closed, \( \cap \)-closed family of subsets of \( X(i) \) with the empty element.

Let \( X \) be an \( n \)-element finite sequence. We introduce MeasurableRectangle \( X \) as a synonym of SemiringProduct \( X \). Now we state the propositions:

(39) Every classical semiring family of \( X \) is a \( \cap \)-closed yielding semiring family of \( X \).

(40) Let us consider a classical semiring family \( S \) of \( X \).
Then MeasurableRectangle \( S \) is a semi-diff-closed, \( \cap \)-closed family of subsets of \( \prod X \) with the empty element. The theorem is a consequence of (39) and (33).

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Two Axiomatizations of Nelson Algebras

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Summary. Nelson algebras were first studied by Rasiowa and Bialynicki-Birula [1] under the name N-lattices or quasi-pseudo-Boolean algebras. Later, in investigations by Monteiro and Brignole [3, 4], and [2] the name “Nelson algebras” was adopted – which is now commonly used to show the correspondence with Nelson’s paper [14] on constructive logic with strong negation.

By a Nelson algebra we mean an abstract algebra
\[ (L, \top, -, \neg, \rightarrow, \Rightarrow, \sqcup, \sqcap) \]
where \( L \) is the carrier, \(-\) is a quasi-complementation (Rasiowa used the sign \( \sim \), but in Mizar “-” should be used to follow the approach described in [12] and [10]), \(-\) is a weak pseudo-complementation, \(\rightarrow\) is weak relative pseudo-complementation and \(\Rightarrow\) is implicative operation. \(\sqcup\) and \(\sqcap\) are ordinary lattice binary operations of supremum and infimum.

In this article we give the definition and basic properties of these algebras according to [16] and [15]. We start with preliminary section on quasi-Boolean algebras (i.e. de Morgan bounded lattices). Later we give the axioms in the form of Mizar adjectives with names corresponding with those in [15]. As our main result we give two axiomatizations (non-equational and equational) and the full formal proof of their equivalence.

The second set of equations is rather long but it shows the logical essence of Nelson lattices. This formalization aims at the construction of algebraic model of rough sets [9] in our future submissions. Section 4 contains all items from Th. 1.2 and 1.3 (and the itemization is given in the text). In the fifth section we provide full formal proof of Th. 2.1 p. 75 [16].

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Keywords: quasi-pseudo-Boolean algebras; Nelson lattices; de Morgan lattices

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The notation and terminology used in this paper have been introduced in the following articles: [5], [6], [7], [18], [11], [13], [17], and [8].

1. De Morgan and Quasi-Boolean Lattices

Let $L$ be a non empty ortholattice structure. We say that $L$ is de Morgan if and only if

(Def. 1) for every elements $x, y$ of $L$, \((x \sqcap y)^c = x^c \sqcup y^c\).

One can verify that every non empty ortholattice structure which is de Morgan and involutive is also de Morgan and every non empty ortholattice structure which is de Morgan and involutive is also de Morgan.

Every non empty ortholattice structure which is trivial is also de Morgan and there exists a non empty ortholattice structure which is de Morgan, involutive, bounded, distributive, and lattice-like.

A de Morgan algebra is a de Morgan, involutive, distributive, lattice-like, non empty ortholattice structure.

A quasi-Boolean algebra is a bounded de Morgan algebra. From now on $L$ denotes a quasi-Boolean algebra and $x, y, z$ denote elements of $L$.

Now we state the propositions:

1. \((x \sqcup y)^c = x^c \sqcap y^c\).
2. \((\top_L)^c = \bot_L\).
3. \((\bot_L)^c = \top_L\).
4. \(x \sqcap (x \sqcap y) = x \sqcap y\).
5. \(x \sqcup (x \sqcup y) = x \sqcup y\).

2. The Structure and Operators in Nelson Algebras

We consider Nelson structures which extend ortholattice structures and are systems

\{(a carrier, a unity, a complement operation, a weak pseudo-complementation, a weak relative pseudo-complementation, an implicative operation, a join operation, a meet operation)\}

where the carrier is a set, the unity is an element of the carrier, the complement operation and the weak pseudo-complementation are unary operations on the carrier, the weak relative pseudo-complementation and the implicative operation
and the join operation and the meet operation are binary operations on the carrier.

Note that there exists a Nelson structure which is strict and non empty and there exists a non empty Nelson structure which is trivial, de Morgan, involutive, bounded, distributive, and lattice-like.

Let $L$ be a non empty Nelson structure and $a, b$ be elements of $L$. The functor $a \rightarrow b$ yielding an element of $L$ is defined by the term

(Def. 2) \((\text{the weak relative pseudo-complementation of } L)(a, b)\).

We say that $a < b$ if and only if

(Def. 3) \(a \rightarrow b = \top_L\).

We say that $a \preceq b$ if and only if

(Def. 4) \(a = a \sqcap b\).

Let $a$ be an element of $L$. The functor $\neg a$ yielding an element of $L$ is defined by the term

(Def. 5) \((\text{the weak pseudo-complementation of } L)(a)\).

Let $a, b$ be elements of $L$. The functor $a \Rightarrow b$ yielding an element of $L$ is defined by the term

(Def. 6) \((\text{the implicative operation of } L)(a, b)\).

3. The Non-Equational Axiomatization

Let $L$ be a non empty Nelson structure. We say that $L$ has reflexive $<$ if and only if

(Def. 7) \(\text{for every element } a \text{ of } L, \ a < a\).

We say that $L$ has transitive $<$ if and only if

(Def. 8) \(\text{for every elements } a, b, c \text{ of } L \text{ such that } a < b < c \text{ holds } a < c\).

Let $L$ be a bounded, lattice-like, non empty Nelson structure. We say that $L$ is quasi-Boolean if and only if

(Def. 9) \(L \text{ is de Morgan, involutive, and distributive}\).

Let us note that every bounded, lattice-like, non empty Nelson structure which is quasi-Boolean is also de Morgan, involutive, and distributive. Every bounded, lattice-like, non empty Nelson structure which is de Morgan, involutive, and distributive is also quasi-Boolean.

Let $L$ be a non empty Nelson structure. We say that $L$ satisfies $(qpB_3)$ if and only if

(Def. 10) \(\text{for every elements } x, a, b \text{ of } L, \ a \sqcap x < b \text{ iff } x < a \rightarrow b\).

We say that $L$ satisfies $(qpB_4)$ if and only if
(Def. 11) for every elements $a, b$ of $L$, $a \Rightarrow b = (a \to b) \cap (\neg b \to \neg a)$.

We say that $L$ satisfies (qpB$_5$) if and only if

(Def. 12) for every elements $a, b$ of $L$, $a \Rightarrow b = \top_L$ iff $a \cap b = a$.

We say that $L$ satisfies (qpB$_6$) if and only if

(Def. 13) for every elements $a, b, c$ of $L$ such that $a < c$ and $b < c$ holds $a \sqcup b < c$.

We say that $L$ satisfies (qpB$_7$) if and only if

(Def. 14) for every elements $a, b, c$ of $L$ such that $a < b$ and $a < c$ holds $a < b \cap c$.

We say that $L$ satisfies (qpB$_8$) if and only if

(Def. 15) for every elements $a, b$ of $L$, $a \sqcup \neg b < -(a \to b)$.

We say that $L$ satisfies (qpB$_9$) if and only if

(Def. 16) for every elements $a, b$ of $L$, $-(a \to b) < a \cap \neg b$.

We say that $L$ satisfies (qpB$_{10}$) if and only if

(Def. 17) for every element $a$ of $L$, $a < \neg \neg a$.

We say that $L$ satisfies (qpB$_{11}$) if and only if

(Def. 18) for every element $a$ of $L$, $\neg \neg a < a$.

We say that $L$ satisfies (qpB$_{12}$) if and only if

(Def. 19) for every elements $a, b$ of $L$, $a \cap \neg a < b$.

We say that $L$ satisfies (qpB$_{13}$) if and only if

(Def. 20) for every element $a$ of $L$, $\neg a = a \to \neg \top_L$.

Let us observe that there exists a bounded, lattice-like, non empty Nelson structure which is quasi-Boolean and has reflexive < and transitive < and satisfies (qpB$_3$), (qpB$_4$), (qpB$_5$), (qpB$_6$), (qpB$_7$), (qpB$_8$), (qpB$_9$), (qpB$_{10}$), (qpB$_{11}$), (qpB$_{12}$), and (qpB$_{13}$).

A Nelson algebra is a quasi-Boolean, bounded, lattice-like, non empty Nelson structure with reflexive < and transitive <. Let $L$ be a bounded, non empty Nelson structure and $a, b$ be elements of $L$. Let us observe that the functor $a \Rightarrow b$ is defined by the term

(Def. 21) $(a \to b) \cap (\neg b \to \neg a)$.

From now on $L$ denotes a Nelson algebra and $a, b, c, d, x, y, z$ denote elements of $L$.

Now we state the propositions:

(6) $a \sqsubseteq b$ if and only if $a \leq b$.

(7) $a \leq b \leq a$ if and only if $a = b$.

**PROOF:** If $a \leq b \leq a$, then $a = b$ by [18, (4), (8)]. □

(8) $a \cap b = \top_L$ if and only if $a = \top_L$ and $b = \top_L$. 


Two axiomatizations of Nelson algebras

(9) $a \leq b$ if and only if $a < b$ and $-b < -a$. The theorem is a consequence of (8).

(10) $a \sqcap b < a$. The theorem is a consequence of (9).

(11) $a < a \sqcup b$. The theorem is a consequence of (9).

(12) $a \leq b$ if and only if $a \Rightarrow b = \top_L$.

(13) $- (a \sqcap b) = -a \sqcup -b$. The theorem is a consequence of (1).

(14) $(a \sqcap -a) \sqcap (b \sqcup -b) = a \sqcap -a$. The theorem is a consequence of (1), (13), and (9).

(15) If $a \leq b \leq c$, then $a \leq c$.

(16) If $b \leq c$, then $a \sqcup b \leq a \sqcup c$ and $a \sqcap b \leq a \sqcap c$. The theorem is a consequence of (9), (1), and (13).

(17) $-a \sqcup b \leq a \Rightarrow b$. The theorem is a consequence of (1), (2), (9), (10), (16), and (15).

(18) $(a \Rightarrow b) \sqcap (-a \sqcap b) = -a \sqcap b$. The theorem is a consequence of (1), (13), (17), (10), (9), and (7).

(19) $-a \sqcup b < a \Rightarrow b$. The theorem is a consequence of (18) and (9).

(20) $a \sqcap (a \Rightarrow b) = a \sqcap (-a \sqcap b)$. The theorem is a consequence of (11), (10), (13), (1), (19), (9), and (7).

(21) If $-x < -y$, then $-(z \rightarrow x) < -(z \rightarrow y)$.

Let us assume that $x < y$. Now we state the propositions:

(22) $a \sqcap (a \Rightarrow x) < y$. The theorem is a consequence of (20) and (10).

(23) $a \Rightarrow x < a \Rightarrow y$. The theorem is a consequence of (22).

(24) $a \Rightarrow (b \sqcap c) = (a \Rightarrow b) \sqcap (a \Rightarrow c)$. The theorem is a consequence of (11), (13), (10), (23), (9), and (7).

4. Properties of Nelson Algebras

Now we state the propositions:

(25) [see also [16] p. 69, Th. 1.2 (5)]:

$a \Rightarrow a = \top_L$.

(26) [see also [16] p. 69, Th. 1.2 (6)]:

If $a \Rightarrow b = \top_L$ and $b \Rightarrow c = \top_L$, then $a \Rightarrow c = \top_L$.

(27) [see also [16] p. 69, Th. 1.2 (7)]:

If $a \Rightarrow b = \top_L$ and $b \Rightarrow a = \top_L$, then $a = b$.

(28) [see also [16] p. 69, Th. 1.2 (8)]:

$a \Rightarrow \top_L = \top_L$. 
(29) [see also 16  P. 69, Th. 1.3 (9)]:
    \[ a \to a = \top_L. \]

(30) [see also 16  P. 69, Th. 1.3 (10)]:
    If \( a \to b = \top_L \) and \( b \to c = \top_L \), then \( a \to c = \top_L \).

(31) [see also 16  P. 69, Th. 1.3 (11)]:
    If \( b < c \), then \( a \mathbin{\lor} b < a \mathbin{\lor} c \) and \( a \mathbin{\land} b < a \mathbin{\land} c \).

(32) [see also 16  P. 69, Th. 1.3 (12)]:
    If \( a < b \) and \( c < d \), then \( a \mathbin{\lor} c < b \mathbin{\lor} d \) and \( a \mathbin{\land} c < b \mathbin{\land} d \).

(33) [see also 16  P. 69, Th. 1.3 (13)]:
    \[ a \mathbin{\land} (a \to b) < b. \]

(34) [see also 16  P. 69, Th. 1.3 (14)]:
    \[ a \to (b \to c) = (a \mathbin{\land} b) \to c. \]

(35) [see also 16  P. 69, Th. 1.3 (15)]:
    \[ a \to (b \to c) = (b \to (a \to c)). \]

(36) [see also 16  P. 69, Th. 1.3 (16)]:
    \[ a < (a \to b) \to b. \] The theorem is a consequence of (33).

(37) [see also 16  P. 71, Th. 1.3 (50)]:
    \[ a < b \to (a \mathbin{\land} b). \] The theorem is a consequence of (9).

(38) [see also 16  P. 69, Th. 1.3 (17)]:
    \[ a \mathbin{\land} -a \leq b \mathbin{\lor} -b. \] The theorem is a consequence of (1) and (9).

(39) [see also 16  P. 70, Th. 1.3 (18)]:
    \[ a \leq b \Rightarrow a \mathbin{\land} b. \] The theorem is a consequence of (37) and (9).

(40) [see also 16  P. 70, Th. 1.3 (19)]:
    \[ a \to \neg b = b \to \neg a. \] The theorem is a consequence of (35).

(41) [see also 16  P. 70, Th. 1.3 (20)]:
    \[ a \to \top_L = \top_L. \] The theorem is a consequence of (9).

(42) [see also 16  P. 70, Th. 1.3 (21)]:
    \[ \bot_L \to a = \top_L. \] The theorem is a consequence of (9).

(43) [see also 16  P. 70, Th. 1.3 (22)]:
    \[ \top_L \to b = b. \] The theorem is a consequence of (9), (33), and (7).

(44) [see also 16  P. 70, Th. 1.3 (23)]:
    If \( a = \top_L \) and \( a \to b = \top_L \), then \( b = \top_L \).

(45) [see also 16  P. 70, Th. 1.3 (24)]:
    \[ a \to (b \to a) = \top_L. \] The theorem is a consequence of (9).

(46) [see also 16  P. 70, Th. 1.3 (25)]:
    \[ (a \to (b \to c)) \to ((a \to b) \to (a \to c)) = \top_L. \] The theorem is a consequence of (33) and (35).
(47) [see also 16 p. 70, Th. 1.3 (26)]:
\[ a \rightarrow (a \sqcup b) = \top_L. \] The theorem is a consequence of (11).

(48) [see also 16 p. 70, Th. 1.3 (27)]:
\[ b \rightarrow (a \sqcup b) = \top_L. \] The theorem is a consequence of (11).

(49) [see also 16 p. 70, Th. 1.3 (28)]:
\[ (a \rightarrow c) \rightarrow ((b \rightarrow c) \rightarrow ((a \sqcup b) \rightarrow c)) = \top_L. \] The theorem is a consequence of (33) and (10).

(50) [see also 16 p. 70, Th. 1.3 (29)]:
\[ (a \cap b) \rightarrow a = \top_L. \] The theorem is a consequence of (10).

(51) [see also 16 p. 70, Th. 1.3 (30)]:
\[ (a \cap b) \rightarrow b = \top_L. \] The theorem is a consequence of (10).

(52) [see also 16 p. 70, Th. 1.3 (31)]:
\[ (a \rightarrow b) \rightarrow ((a \rightarrow c) \rightarrow (a \rightarrow (b \cap c))) = \top_L. \] The theorem is a consequence of (33).

(53) [see also 16 p. 70, Th. 1.3 (32)]:
\[ (a \rightarrow \neg b) \rightarrow (b \rightarrow \neg a) = \top_L. \] The theorem is a consequence of (40) and (29).

(54) [see also 16 p. 70, Th. 1.3 (33)]:
\[ \neg (a \rightarrow a) \rightarrow b = \top_L. \] The theorem is a consequence of (29), (2), (43), and (42).

(55) [see also 16 p. 70, Th. 1.3 (34)]:
\[ \neg a \rightarrow (a \rightarrow b) = \top_L. \]

(56) [see also 16 p. 70, Th. 1.3 (35)]:
\[ (\neg (a \rightarrow b) \rightarrow (a \cap \neg b)) \cap ((a \cap \neg b) \rightarrow \neg (a \rightarrow b)) = \top_L. \]

(57) [see also 16 p. 70, Th. 1.3 (36)]:
\[ (\neg a \rightarrow a) \cap (a \rightarrow \neg a) = \top_L. \]

(58) [see also 16 p. 70, Th. 1.3 (37)]:
\[ \neg \neg a = a. \]

(59) [see also 16 p. 70, Th. 1.3 (38)]:
\[ (a \sqcup b) = \neg a \cap \neg b. \]

(60) [see also 16 p. 70, Th. 1.3 (39)]:
\[ (a \cap b) = \neg a \cap \neg b. \] The theorem is a consequence of (1).

(61) [see also 16 p. 70, Th. 1.3 (40)]:
If \( a < b \), then \( b \rightarrow c < a \rightarrow c \) and \( c \rightarrow a < c \rightarrow b \). The theorem is a consequence of (43), (46), (10), and (41).

(62) [see also 16 p. 70, Th. 1.3 (41)]:
\[ (a \rightarrow b) \rightarrow ((c \rightarrow d) \rightarrow ((a \cap c) \rightarrow (b \cap d))) = \top_L. \] The theorem is a consequence of (33).
(63) [see also [16] p. 70, Th. 1.3 (42)]:
\[(a \rightarrow b) \rightarrow ((c \rightarrow d) \rightarrow ((a \sqcup c) \rightarrow (b \sqcup d))) = \top_L.\]
The theorem is a consequence of (10).

(64) [see also [16] p. 70, Th. 1.3 (43)]:
\[(b \rightarrow a) \rightarrow ((c \rightarrow d) \rightarrow ((a \rightarrow c) \rightarrow (b \rightarrow d))) = \top_L.\]
The theorem is a consequence of (33).

5. Alternative Equational Axiomatics by Rasiowa

Let \( L \) be a non empty Nelson structure. We say that \( L \) satisfies \( (qPB_0^*) \) if and only if

(Def. 22) for every elements \( a, b \) of \( L \), \( a \preceq b \) iff \( a \Rightarrow b = \top_L. \)

We say that \( L \) satisfies \( (qPB_1^*) \) if and only if

(Def. 23) for every elements \( a, b \) of \( L \), \( a \rightarrow (b \rightarrow a) = \top_L. \)

We say that \( L \) satisfies \( (qPB_2^*) \) if and only if

(Def. 24) for every elements \( a, b, c \) of \( L \), \( (a \rightarrow (b \rightarrow c)) \rightarrow ((a \rightarrow b) \rightarrow (a \rightarrow c)) = \top_L. \)

We say that \( L \) satisfies \( (qPB_3^*) \) if and only if

(Def. 25) for every element \( a \) of \( L \), \( \top_L \rightarrow a = a. \)

We say that \( L \) satisfies \( (qPB_5^*) \) if and only if

(Def. 26) for every elements \( a, b \) of \( L \), \( (a \Rightarrow b) \rightarrow ((b \Rightarrow a) \rightarrow b) = (b \Rightarrow a) \rightarrow ((a \Rightarrow b) \rightarrow a). \)

We say that \( L \) satisfies \( (qPB_6^*) \) if and only if

(Def. 27) for every elements \( a, b \) of \( L \), \( a \rightarrow (a \sqcup b) = \top_L. \)

We say that \( L \) satisfies \( (qPB_7^*) \) if and only if

(Def. 28) for every elements \( a, b \) of \( L \), \( b \rightarrow (a \sqcup b) = \top_L. \)

We say that \( L \) satisfies \( (qPB_8^*) \) if and only if

(Def. 29) for every elements \( a, b, c \) of \( L \), \( (a \rightarrow c) \rightarrow ((b \rightarrow c) \rightarrow ((a \sqcup b) \rightarrow c)) = \top_L. \)

We say that \( L \) satisfies \( (qPB_9^*) \) if and only if

(Def. 30) for every elements \( a, b \) of \( L \), \( (a \sqcap b) \rightarrow a = \top_L. \)

We say that \( L \) satisfies \( (qPB_{10}^*) \) if and only if

(Def. 31) for every elements \( a, b \) of \( L \), \( (a \sqcap b) \rightarrow b = \top_L. \)

We say that \( L \) satisfies \( (qPB_{11}^*) \) if and only if

(Def. 32) for every elements \( a, b, c \) of \( L \), \( (a \rightarrow b) \rightarrow ((a \rightarrow c) \rightarrow (a \rightarrow (b \sqcap c))) = \top_L. \)

We say that \( L \) satisfies \( (qPB_{12}^*) \) if and only if

(Def. 33) for every elements \( a, b \) of \( L \), \( (a \rightarrow \neg b) \rightarrow (b \rightarrow \neg a) = \top_L. \)

We say that \( L \) satisfies \( (qPB_{13}^*) \) if and only if
(Def. 34) for every elements \(a, b\) of \(L\), \(\neg(a \rightarrow a) \rightarrow b = \top_L\).

We say that \(L\) satisfies (qpB\(^+_1\)) if and only if

(Def. 35) for every elements \(a, b\) of \(L\), \(-a \rightarrow (a \rightarrow b) = \top_L\).

We say that \(L\) satisfies (qpB\(^+_5\)) if and only if

(Def. 36) for every elements \(a, b\) of \(L\), \((a \rightarrow b) \cap (a \rightarrow \neg b) = (a \rightarrow \neg (a \rightarrow b)) = \top_L\).

We say that \(L\) satisfies (qpB\(^+_7\)) if and only if

(Def. 37) for every elements \(a, b\) of \(L\), \(-a \cup b = -a \cap b\).

We say that \(L\) satisfies (qpB\(^+_9\)) if and only if

(Def. 38) for every element \(a\) of \(L\), \((\neg a \rightarrow a) \cap (a \rightarrow \neg a) = \top_L\).

We introduce \(L\) satisfies (qpB\(^+_k\)) as a synonym of \(L\) satisfies (qpB\(_4\)) and \(L\) satisfies (qpB\(_{16}\)) as a synonym of \(L\) is de Morgan and \(L\) satisfies (qpB\(_{18}\)) as a synonym of \(L\) is involutive.

Note that every Nelson algebra satisfies (qpB\(_1\)), (qpB\(_2\)), (qpB\(_3\)), (qpB\(_4\)), (qpB\(_5^*\)), (qpB\(_6^*\)), (qpB\(_7^*\)), (qpB\(_8^*\)), (qpB\(_9^*\)), (qpB\(_{10}^*\)), (qpB\(_{11}^*\)), (qpB\(_{12}^*\)), (qpB\(_{13}^*\)), (qpB\(_{14}^*\)), (qpB\(_{15}^*\)), (qpB\(_{16}^*\)), (qpB\(_{17}^*\)), (qpB\(_{18}^*\)), and (qpB\(_{19}^*\)).

Now we state the proposition:

(65) Let us consider a non empty Nelson structure \(L\). Suppose \(L\) satisfies (qpB\(_0^*\)). Then \(L\) is a Nelson algebra if and only if \(L\) satisfies (qpB\(_1^*\)), (qpB\(_2^*\)), (qpB\(_3^*\)), (qpB\(_4^*\)), (qpB\(_5^*\)), (qpB\(_6^*\)), (qpB\(_7^*\)), (qpB\(_8^*\)), (qpB\(_9^*\)), (qpB\(_{10}^*\)), (qpB\(_{11}^*\)), (qpB\(_{12}^*\)), (qpB\(_{13}^*\)), (qpB\(_{14}^*\)), (qpB\(_{15}^*\)), (qpB\(_{16}^*\)), (qpB\(_{17}^*\)), (qpB\(_{18}^*\)), and (qpB\(_{19}^*\)).

Proof: Reconsider \(L_1 = L\) as a de Morgan, non empty Nelson structure. For every elements \(a, b\) of \(L_1\) such that \(a \cap b = \top_{L_1}\) holds \(a = \top_{L_1}\) and \(b = \top_{L_1}\).

For every elements \(a, b\) of \(L_1\), \(a \leq b\) iff \(a < b\) and \(-b < -a\). Set \(d = (\top_{L_1})^c\). For every element \(a\) of \(L_1\), \(d \leq a\).

For every element \(a\) of \(L_1\), \(d \cap a = d\).

For every element \(a\) of \(L_1\), \(a \rightarrow \top_{L_1} = \top_{L_1}\).

For every elements \(a, b, c\) of \(L_1\) such that \(a \rightarrow b = \top_{L_1}\) and \(b \rightarrow c = \top_{L_1}\) holds \(a \rightarrow c = \top_{L_1}\).

\(L_1\) has transitive \(<\).

\(L_1\) satisfies (qpB\(_6^*\)).

For every element \(a\) of \(L_1\), \(a \rightarrow a = \top_{L_1}\).

\(L_1\) satisfies (qpB\(_7^*\)).

For every elements \(a, b\) of \(L_1\), \(a \cap b \leq a\).

For every elements \(a, b\) of \(L_1\), \(a \leq a \cap b\).

For every elements \(a, b\) of \(L_1\), \(a \cap b \leq b\).

For every element \(a\) of \(L_1\), \(a \rightarrow a = \top_{L_1}\).

For every elements \(a, b\) of \(L_1\), \(a \Rightarrow b = \top_{L_1}\) and \(b \Rightarrow a = \top_{L_1}\).

For every elements \(a, b\) of \(L_1\), \(a \leq b \leq a\).

For every elements \(a, b\) of \(L_1\) such that \(a \leq b\) holds \(b \rightarrow c < b \rightarrow c\) and \(c \rightarrow a < c \rightarrow b\).

For every elements \(a, b\) of \(L_1\), \(a \rightarrow b \rightarrow (a \cap b) = \top_{L_1}\).

For every elements \(a, b, c\) of \(L_1\) such that \(a \leq b \rightarrow c\) holds \(b \rightarrow c \rightarrow a < a \rightarrow c\).

For every elements \(a, b, c\) of \(L_1\), \(a \rightarrow (a \rightarrow c) < a \rightarrow c\).

\(L_1\) satisfies (qpB\(_3^*\)).

For every elements \(a, b, c\) of \(L_1\)
such that $b < c$ holds $a \cap b < a \cap c$. For every elements $a$, $b$, $c$ of $L_1$ such that $b < c$ holds $a \sqcup b < a \sqcup c$. For every elements $a$, $b$, $c$ of $L_1$ such that $a \leq c$ and $b \leq c$ holds $a \sqcup b \leq c$. For every elements $a$, $b$, $c$ of $L_1$ such that $c \leq a$ and $c \leq b$ holds $c \leq a \cap b$. For every elements $a$, $b$ of $L_1$, $b \sqcup a \leq a \sqcup b$. For every elements $a$, $b$ of $L_1$, $a \sqcup b \leq b \sqcup a$. For every elements $a$, $b$, $c$ of $L_1$ such that $a \leq b$ holds $a \sqcup c \leq b \sqcup c$. For every elements $a$, $b$ of $L_1$, $a \cap (a \sqcup b) = a$. For every elements $a$, $b$, $c$ of $L_1$ such that $b < c$ holds $a \sqcap b \leq a \sqcap c$. For every elements $a$, $b$, $c$ of $L_1$ such that $a \leq b$ holds $a \leq c$. For every elements $a$, $b$, $c$ of $L_1$, $a \sqcap (b \sqcup c) = (a \sqcap b) \sqcap c$. For every elements $a$, $b$, $c$ of $L_1$, $a \sqcup (b \sqcup c) = (a \sqcup b) \sqcup c$. Set $c = \top L_1$. For every element $a$ of $L_1$, $c \sqcup a = c$ and $a \sqcup c = c$ by [18, (4)]. $L_1$ is distributive. $L_1$ satisfies $(qbB_5)$. $L_1$ satisfies $(qbB_8)$. $L_1$ satisfies $(qbB_9)$. $L_1$ satisfies $(qbB_{10})$. $L_1$ satisfies $(qbB_{11})$. $L_1$ satisfies $(qbB_{12})$. For every elements $a$, $b$, $c$ of $L_1$, $\neg \top L_1 = \neg \top L_1$. For every elements $a$, $b$ of $L_1$, $a \rightarrow \neg b = b \rightarrow \neg a$. $L_1$ satisfies $(qbB_{13})$. □

\section*{References}


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Summary. We translate the articles covering group theory already available in the Mizar Mathematical Library from multiplicative into additive notation. We adapt the works of Wojciech A. Trybulec \cite{41,42,43} and Artur Korniłowicz \cite{25}.

In particular, these authors have defined the notions of group, abelian group, power of an element of a group, order of a group and order of an element, subgroup, coset of a subgroup, index of a subgroup, conjugation, normal subgroup, topological group, dense subset and basis of a topological group. Lagrange’s theorem and some other theorems concerning these notions \cite{9,21,22} are presented.

Note that “The term $\mathbb{Z}$-module is simply another name for an additive abelian group” \cite{27}. We take an approach different than that used by Futa et al. \cite{21} to use in a future article the results obtained by Artur Korniłowicz \cite{25}. Indeed, Hölzl et al. showed that it was possible to build “a generic theory of limits based on filters” in Isabelle/HOL \cite{23,10}. Our goal is to define the convergence of a sequence and the convergence of a series in an abelian topological group \cite{11} using the notion of filters.

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Keywords: additive group; subgroup; Lagrange theorem; conjugation; normal subgroup; index; additive topological group; basis; neighborhood; additive abelian group; $\mathbb{Z}$-module

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The notation and terminology used in this paper have been introduced in the following articles: \cite{12}, \cite{32}, \cite{31}, \cite{2}, \cite{18}, \cite{28}, \cite{33}, \cite{13}, \cite{19}, \cite{39}, \cite{14}, \cite{15}, \cite{1}, \cite{40}, \cite{26}, \cite{35}, \cite{36}, \cite{5}, \cite{6}, \cite{16}, \cite{30}, \cite{8}, \cite{46}, \cite{47}, \cite{44}, \cite{29}, \cite{37}, \cite{45}, \cite{25}, \cite{48}, \cite{20}, \cite{7}, \cite{38}, and \cite{17}.
From now on $m$, $n$ denote natural numbers, $i$, $j$ denote integers, $S$ denotes a non empty additive magma, and $r$, $r_1$, $r_2$, $s$, $s_1$, $s_2$, $t$, $t_1$, $t_2$ denote elements of $S$.

The scheme $SeqEx2Dbis$ deals with non empty sets $X$, $Z$ and a ternary predicate $P$ and states that

(Sch. 1) There exists a function $f$ from $\mathbb{N} \times X$ into $Z$ such that for every natural number $x$ for every element $y$ of $X$, $P[x, y, f(x, y)]$

provided

- for every natural number $x$ and for every element $y$ of $X$, there exists an element $z$ of $Z$ such that $P[x, y, z]$.

Let $I_1$ be an additive magma. We say that $I_1$ is add-unital if and only if

(Def. 1) there exists an element $e$ of $I_1$ such that for every element $h$ of $I_1$, $h + e = h$ and $e + h = h$.

We say that $I_1$ is additive group-like if and only if

(Def. 2) there exists an element $e$ of $I_1$ such that for every element $h$ of $I_1$, $h + e = h$ and $e + h = h$ and there exists an element $g$ of $I_1$ such that $h + g = e$ and $g + h = e$.

Let us note that every additive magma which is additive group-like is also add-unital and there exists an additive magma which is strict, additive group-like, add-associative, and non empty.

An additive group is an additive group-like, add-associative, non empty additive magma. Now we state the propositions:

(1) Suppose for every $r$, $s$, and $t$, $(r + s) + t = r + (s + t)$ and there exists $t$ such that for every $s_1$, $s_1 + t = s_1$ and $t + s_1 = s_1$ and there exists $s_2$ such that $s_1 + s_2 = t$ and $s_2 + s_1 = t$. Then $S$ is an additive group.

(2) Suppose for every $r$, $s$, and $t$, $(r + s) + t = r + (s + t)$ and for every $r$ and $s$, there exists $t$ such that $r + t = s$ and there exists $t$ such that $t + r = s$. Then $S$ is add-associative and additive group-like.

(3) $\langle \mathbb{R}, + \rangle$ is add-associative and additive group-like.

From now on $G$ denotes an additive group-like, non empty additive magma and $e$, $h$ denote elements of $G$.

Let $G$ be an additive magma. Assume $G$ is add-unital. The functor $0_G$ yielding an element of $G$ is defined by

(Def. 3) for every element $h$ of $G$, $h + it = h$ and $it + h = h$.

Now we state the proposition:
(4) If for every \( h, \ h + e = h \) and \( e + h = h \), then \( e = 0_G \).

From now on \( G \) denotes an additive group and \( f, g, h \) denote elements of \( G \).

Let us consider \( G \) and \( h \). The functor \(-h\) yielding an element of \( G \) is defined by

(Def. 4) \( h + it = 0_G \) and \( it + h = 0_G \).

Let us note that the functor is involutive.

Now we state the propositions:

(5) If \( h + g = 0_G \) and \( g + h = 0_G \), then \( g = -h \).

(6) If \( h + g = h + f \) or \( g + h = f + h \), then \( g = f \).

(7) If \( h + g = h \) or \( g + h = h \), then \( g = 0_G \). The theorem is a consequence of (6).

(8) \(-0_G = 0_G\).

(9) If \( -h = -g \), then \( h = g \). The theorem is a consequence of (6).

(10) If \( -h = 0_G \), then \( h = 0_G \). The theorem is a consequence of (8).

(11) If \( h + g = 0_G \), then \( h = -g \) and \( g = -h \). The theorem is a consequence of (6).

(12) \( h + f = g \) if and only if \( f = -h + g \). The theorem is a consequence of (6).

(13) \( f + h = g \) if and only if \( f = g + -h \). The theorem is a consequence of (6).

(14) There exists \( f \) such that \( g + f = h \). The theorem is a consequence of (12).

(15) There exists \( f \) such that \( f + g = h \). The theorem is a consequence of (13).

(16) \(- (h + g) = -g + -h \). The theorem is a consequence of (11).

(17) \( g + h = h + g \) if and only if \( -(g + h) = -g + -h \). The theorem is a consequence of (16) and (6).

(18) \( g + h = h + g \) if and only if \( -g + -h = -h + g \). The theorem is a consequence of (16) and (17).

(19) \( g + h = h + g \) if and only if \( g + -h = -h + g \). The theorem is a consequence of (18), (11), and (16).

From now on \( u \) denotes a unary operation on \( G \).

Let us consider \( G \). The functor add inverse \( G \) yielding a unary operation on \( G \) is defined by

(Def. 5) \( u(h) = -h \).

Let \( G \) be an add-associative, non empty additive magma. Let us note that the addition of \( G \) is associative.
Let us consider an add-unital, non empty additive magma $G$. Now we state the propositions:

(20) $0_G$ is a unity w.r.t. the addition of $G$.

(21) $1_\alpha = 0_G$, where $\alpha$ is the addition of $G$. The theorem is a consequence of (20).

Let $G$ be an add-unital, non empty additive magma. Let us note that the addition of $G$ is unital.

Now we state the proposition:

(22) add inverse $G$ is an inverse operation w.r.t. the addition of $G$. The theorem is a consequence of (21).

Let us consider $G$. One can verify that the addition of $G$ has inverse operation.

Now we state the proposition:

(23) The inverse operation w.r.t. the addition of $G = \text{add inverse } G$. The theorem is a consequence of (22).

Let $G$ be a non empty additive magma. The functor \(\text{mult } G\) yielding a function from $\mathbb{N} \times (\text{the carrier of } G)$ into the carrier of $G$ is defined by

(Def. 6) for every element $h$ of $G$, \(it(0, h) = 0_G\) and for every natural number $n$,
\[
\text{it}(n + 1, h) = \text{it}(n, h) + h.
\]

Let us consider $G$, $i$, and $h$. The functor $i \cdot h$ yielding an element of $G$ is defined by the term

(Def. 7) \[
\begin{cases} 
(\text{mult } G)(|i|, h), & \text{if } 0 \leq i, \\
-(\text{mult } G)(|i|, h), & \text{otherwise.} 
\end{cases}
\]

Let us consider $n$. One can check that the functor $n \cdot h$ is defined by the term

(Def. 8) \[
(\text{mult } G)(n, h).
\]

Now we state the propositions:

(24) $0 \cdot h = 0_G$.

(25) $1 \cdot h = h$.

(26) $2 \cdot h = h + h$. The theorem is a consequence of (25).

(27) $3 \cdot h = h + h + h$. The theorem is a consequence of (26).

(28) $2 \cdot h = 0_G$ if and only if $-h = h$. The theorem is a consequence of (26) and (11).

(29) If $i \leq 0$, then $i \cdot h = -|i| \cdot h$. The theorem is a consequence of (8).

(30) $i \cdot 0_G = 0_G$. The theorem is a consequence of (8).

(31) $(-1) \cdot h = -h$. The theorem is a consequence of (25).

(32) $(i + j) \cdot h = i \cdot h + j \cdot h$. 

Proof: Define $P[\text{integer}]$ for every $i$, $(i + 1) \cdot h = i \cdot h + \$1 \cdot h$. For every $j$ such that $P[j]$ holds $P[j - 1]$ and $P[j + 1]$. $P[0]$. For every $j$, $P[j]$ from [40]. Sch. 4]. □

(33) (i) $(i + 1) \cdot h = i \cdot h + h$, and
(ii) $(i + 1) \cdot h = h + i \cdot h$.

The theorem is a consequence of (25) and (32).

(34) $(-i) \cdot h = -i \cdot h$.

Let us assume that $g + h = h + g$. Now we state the propositions:

(35) $i \cdot (g + h) = i \cdot g + i \cdot h$. The theorem is a consequence of (16).

(36) $i \cdot g + j \cdot h = j \cdot h + i \cdot g$. The theorem is a consequence of (19) and (16).

(37) $g + i \cdot h = i \cdot h + g$. The theorem is a consequence of (25) and (36).

Let us consider $G$ and $h$. We say that $h$ is of order 0 if and only if

(Def. 9) if $n \cdot h = 0_G$, then $n = 0$.

One can check that $0_G$ is non of order 0.

Let us consider $h$. The functor ord($h$) yielding an element of $\mathbb{N}$ is defined by

(Def. 10) (i) $it = 0$, if $h$ is of order 0,
(ii) $it \cdot h = 0_G$ and $it \neq 0$ and for every $m$ such that $m \cdot h = 0_G$ and $m \neq 0$ holds $it \leq m$, otherwise.

Now we state the propositions:

(38) $\text{ord}(h) \cdot h = 0_G$.

(39) $\text{ord}(0_G) = 1$.

(40) If $\text{ord}(h) = 1$, then $h = 0_G$. The theorem is a consequence of (25).

Observe that there exists an additive group which is strict and Abelian.

Now we state the proposition:

(41) $\langle \mathbb{R}, +_{\mathbb{R}} \rangle$ is an Abelian additive group. The theorem is a consequence of (3).

In the sequel $A$ denotes an Abelian additive group and $a, b$ denote elements of $A$.

Now we state the propositions:

(42) $-(a + b) = -a + -b$.

(43) $i \cdot (a + b) = i \cdot a + i \cdot b$.

(44) $\langle$ the carrier of $A$, the addition of $A, 0_A \rangle$ is Abelian, add-associative, right zeroed, and right complementable.

Let us consider an add-unital, non empty additive magma $L$ and an element $x$ of $L$. Now we state the propositions:

(45) $(\text{mult} L)(1, x) = x$. 


(46) \((\text{mult } L)(2, x) = x + x\). The theorem is a consequence of (45).

Now we state the proposition:

(47) Let us consider an add-associative, Abelian, add-unital, non empty additive magma \(L\), elements \(x, y\) of \(L\), and a natural number \(n\). Then 
\((\text{mult } L)(n, x + y) = (\text{mult } L)(n, x) + (\text{mult } L)(n, y)\).

**Proof:** Define \(P[\text{natural number}] \equiv (\text{mult } L)(1, x + y) = (\text{mult } L)(1, x) + (\text{mult } L)(1, y)\). For every natural number \(n\), \(P[n]\) from [5, Sch. 2]. \(\square\)

Let \(G, H\) be additive magmas and \(I_1\) be a function from \(G\) into \(H\). We say that \(I_1\) preserves zero if and only if

(Def. 11) \(I_1(0_G) = 0_H\).

2. **Subgroups and Lagrange Theorem — GROUP 2**

In the sequel \(x\) denotes an object, \(y, y_1, y_2, Y, Z\) denote sets, \(k\) denotes a natural number, \(G\) denotes an additive group, \(a, g, h\) denote elements of \(G\), and \(A\) denotes a subset of \(G\).

Let us consider \(G\) and \(A\). The functor \(-A\) yielding a subset of \(G\) is defined by the term 

(Def. 12) \(\{-g : g \in A\}\).

One can check that the functor is involutive.

Now we state the propositions:

(48) \(x \in -A\) if and only if there exists \(g\) such that \(x = -g\) and \(g \in A\).

(49) \(-\{g\} = \{-g\}\).

(50) \(-\{g, h\} = \{-g, -h\}\).

(51) \(-\emptyset_\alpha = \emptyset, \text{ where } \alpha \text{ is the carrier of } G\).

(52) \(-\Omega_\alpha = \text{ the carrier of } G, \text{ where } \alpha \text{ is the carrier of } G\).

(53) \(A \neq \emptyset\) if and only if \(-A \neq \emptyset\). The theorem is a consequence of (48).

Let us consider \(G\). Let \(A\) be an empty subset of \(G\). Observe that \(-A\) is empty.

Let \(A\) be a non empty subset of \(G\). One can check that \(-A\) is non empty.

In the sequel \(G\) denotes a non empty additive magma, \(A, B, C\) denote subsets of \(G\), and \(a, b, g, g_1, g_2, h, h_1, h_2\) denote elements of \(G\).

Let \(G\) be an Abelian, non empty additive magma and \(A, B\) be subsets of \(G\). One can check that the functor \(A + B\) is commutative.

(54) \(x \in A + B\) if and only if there exists \(g\) and there exists \(h\) such that \(x = g + h\) and \(g \in A\) and \(h \in B\).

(55) \(A \neq \emptyset\) and \(B \neq \emptyset\) if and only if \(A + B \neq \emptyset\). The theorem is a consequence of (54).
(56) If $G$ is add-associative, then $(A + B) + C = A + (B + C)$.
(57) Let us consider an additive group $G$, and subsets $A, B$ of $G$. Then 
$-(A + B) = -B + -A$. The theorem is a consequence of (16).

(58) $A + (B \cup C) = A + B \cup (A + C)$.

(59) $A + B \cap C \subseteq (A + B) \cap (A + C)$.

(60) $A \cap B + C \subseteq (A + C) \cap (B + C)$.

(61) $A + B \setminus C \subseteq (A + B) \setminus (A + C)$.

(62) (i) $\emptyset + A = \emptyset$, and 
(ii) $A + \emptyset = \emptyset$,
where $\alpha$ is the carrier of $G$. The theorem is a consequence of (54).

(63) Let us consider an additive group $G$, and a subset $A$ of $G$. Suppose $A \neq \emptyset$. Then 
(i) $\Omega_{\alpha} + A$ is the carrier of $G$, and 
(ii) $A + \Omega_{\alpha}$ is the carrier of $G$,
where $\alpha$ is the carrier of $G$.

(64) $\{g\} + \{h\} = \{g + h\}$.

(65) $\{g\} + \{g_1, g_2\} = \{g + g_1, g + g_2\}$.

(66) $\{g_1, g_2\} + \{g\} = \{g_1 + g, g_2 + g\}$.

(67) $\{g, h\} + \{g_1, g_2\} = \{g + g_1, g + g_2, h + g_1, h + g_2\}$.

Let us consider an additive group $G$ and a subset $A$ of $G$. Now we state the
propositions:

(68) Suppose for every elements $g_1, g_2$ of $G$ such that $g_1, g_2 \in A$ holds $g_1 + g_2 \in A$ 
and for every element $g$ of $G$ such that $g \in A$ holds $-g \in A$. Then 
$A + A = A$.

(69) If for every element $g$ of $G$ such that $g \in A$ holds $-g \in A$, then $-A = A$.

(70) If for every $a$ and $b$ such that $a \in A$ and $b \in B$ holds $a + b = b + a$, then 
$A + B = B + A$.

(71) If $G$ is an Abelian additive group, then $A + B = B + A$.

(72) Let us consider an Abelian additive group $G$, and subsets $A, B$ of $G$. 
Then $-(A + B) = -A + -B$. The theorem is a consequence of (42).

Let us consider $G, g$, and $A$. The functors: $g + A$ and $A + g$ yielding subsets 
of $G$ are defined by terms,

(Def. 13) $\{g\} + A$,

(Def. 14) $A + \{g\}$,
respectively. Now we state the propositions:

(73) $x \in g + A$ if and only if there exists $h$ such that $x = g + h$ and $h \in A$. 

Let us assume that $G$ is add-associative. Now we state the propositions:

(75) \((g + A) + B = g + (A + B)\).

(76) \((A + g) + B = A + (g + B)\).

(77) \((A + B) + g = A + (B + g)\).

(78) \((g + h) + A = g + (h + A)\). The theorem is a consequence of (64) and (56).

(79) \((g + A) + h = g + (A + h)\).

(80) \((A + g) + h = A + (g + h)\). The theorem is a consequence of (56) and (64).

(81) (i) \(\emptyset \alpha + a = \emptyset\), and

(ii) \(a + \emptyset \alpha = \emptyset\),

where $\alpha$ is the carrier of $G$.

From now on $G$ denotes an additive group-like, non empty additive magma, $h, g, g_1, g_2$ denote elements of $G$, and $A$ denotes a subset of $G$.

(82) Let us consider an additive group $G$, and an element $a$ of $G$. Then

(i) \(\Omega \alpha + a = \text{the carrier of } G\), and

(ii) \(a + \Omega \alpha = \text{the carrier of } G\),

where $\alpha$ is the carrier of $G$.

(83) (i) \(0_G + A = A\), and

(ii) \(A + 0_G = A\).

The theorem is a consequence of (73) and (74).

(84) If $G$ is an Abelian additive group, then $g + A = A + g$.

Let $G$ be an additive group-like, non empty additive magma.

A subgroup of $G$ is an additive group-like, non empty additive magma and is defined by

(Def. 15) the carrier of $it \subseteq$ the carrier of $G$ and the addition of $it = (\text{the addition of } G) \upharpoonright (\text{the carrier of } it)$.

In the sequel $H$ denotes a subgroup of $G$ and $h, h_1, h_2$ denote elements of $H$.

Now we state the propositions:

(85) If $G$ is finite, then $H$ is finite.

(86) If $x \in H$, then $x \in G$.

(87) $h \in G$.

(88) $h$ is an element of $G$.

(89) If $h_1 = g_1$ and $h_2 = g_2$, then $h_1 + h_2 = g_1 + g_2$. 
Let $G$ be an additive group. Let us observe that every subgroup of $G$ is add-associative.

In the sequel $G, G_1, G_2, G_3$ denote additive groups, $a, a_1, a_2, b, b_1, b_2, g, g_1, g_2$ denote elements of $G$, $A, B$ denote subsets of $G$, $H, H_1, H_2, H_3$ denote subgroups of $G$, and $h, h_1, h_2$ denote elements of $H$.

(90) $0_H = 0_G$. The theorem is a consequence of (87), (89), and (7).

(91) $0_{H_1} = 0_{H_2}$. The theorem is a consequence of (90).

(92) $0_G \in H$. The theorem is a consequence of (90).

(93) $0_{H_1} \in H_2$. The theorem is a consequence of (90) and (92).

(94) If $h = g$, then $-h = -g$. The theorem is a consequence of (87), (89), (90), and (11).

(95) add inverse $H = \text{add inverse } G|\text{(the carrier of } H)$. The theorem is a consequence of (87) and (94).

(96) If $g_1, g_2 \in H$, then $g_1 + g_2 \in H$. The theorem is a consequence of (89).

(97) If $g \in H$, then $-g \in H$. The theorem is a consequence of (94).

Let us consider $G$. Observe that there exists a subgroup of $G$ which is strict.

(98) Suppose $A \neq \emptyset$ and for every $g_1$ and $g_2$ such that $g_1, g_2 \in A$ holds $g_1 + g_2 \in A$ and for every $g$ such that $g \in A$ holds $-g \in A$. Then there exists a strict subgroup $H$ of $G$ such that the carrier of $H = A$.

**Proof:** Reconsider $D = A$ as a non empty set. Set $o = (\text{the addition of } G) | A$. $\text{rng } o \subseteq A$ by [17], (87)], [14], (47)]. Set $H = \langle D, o \rangle$. $H$ is additive group-like. □

(99) If $G$ is an Abelian additive group, then $H$ is Abelian. The theorem is a consequence of (87) and (89).

Let $G$ be an Abelian additive group. One can check that every subgroup of $G$ is Abelian.

(100) $G$ is a subgroup of $G$.

(101) Suppose $G_1$ is a subgroup of $G_2$ and $G_2$ is a subgroup of $G_1$. Then the additive magma of $G_1 = \text{the additive magma of } G_2$.

(102) If $G_1$ is a subgroup of $G_2$ and $G_2$ is a subgroup of $G_3$, then $G_1$ is a subgroup of $G_3$.

(103) If the carrier of $H_1 \subseteq \text{the carrier of } H_2$, then $H_1$ is a subgroup of $H_2$.

(104) If for every $g$ such that $g \in H_1$ holds $g \in H_2$, then $H_1$ is a subgroup of $H_2$. The theorem is a consequence of (87) and (103).

(105) Suppose the carrier of $H_1 = \text{the carrier of } H_2$. Then the additive magma of $H_1 = \text{the additive magma of } H_2$. The theorem is a consequence of (103) and (101).
(106) Suppose for every $g$, $g \in H_1$ iff $g \in H_2$. Then the additive magma of $H_1 = \text{the additive magma of } H_2$. The theorem is a consequence of (104) and (101).

Let us consider $G$. Let $H_1$, $H_2$ be strict subgroups of $G$. One can check that $H_1 = H_2$ if and only if the condition (Def. 16) is satisfied.

(Def. 16) for every $g$, $g \in H_1$ iff $g \in H_2$.

Now we state the propositions:

(107) Let us consider an additive group $G$, and a subgroup $H$ of $G$. Suppose the carrier of $G \subseteq$ the carrier of $H$. Then the additive magma of $H = \text{the additive magma of } G$. The theorem is a consequence of (100) and (105).

(108) Suppose for every element $g$ of $G$, $g \in H$. Then the additive magma of $H = \text{the additive magma of } G$. The theorem is a consequence of (100) and (106).

Let us consider $G$. The functor $0_G$ yielding a strict subgroup of $G$ is defined by

(Def. 17) the carrier of $it = \{0_G\}$.

The functor $\Omega_G$ yielding a strict subgroup of $G$ is defined by the term

(Def. 18) the additive magma of $G$.

Note that the functor is projective.

Now we state the propositions:

(109) $0_H = 0_G$. The theorem is a consequence of (90) and (102).

(110) $0_{H_1} = 0_{H_2}$. The theorem is a consequence of (109).

(111) $0_G$ is a subgroup of $H$. The theorem is a consequence of (109).

(112) Let us consider a strict additive group $G$. Then every subgroup of $G$ is a subgroup of $\Omega_G$.

(113) Every strict additive group is a subgroup of $\Omega_G$.

(114) $0_G$ is finite.

Let us consider $G$. Note that $0_G$ is finite and there exists a subgroup of $G$ which is strict and finite and there exists an additive group which is strict and finite.

Let $G$ be a finite additive group. One can verify that every subgroup of $G$ is finite.

Now we state the propositions:

(115) $\overline{0}_G = 1$.

(116) Let us consider a strict, finite subgroup $H$ of $G$. If $\overline{H} = 1$, then $H = 0_G$. The theorem is a consequence of (92).
Groups – additive notation

(117) \( \overline{H} \subseteq \overline{G} \).

Let us consider a finite additive group \( G \) and a subgroup \( H \) of \( G \). Now we state the propositions:

(118) \( \overline{H} \leq \overline{G} \).

(119) Suppose \( \overline{G} = \overline{H} \). Then the additive magma of \( H \) = the additive magma of \( G \).

Proof: The carrier of \( H \) = the carrier of \( G \) by \[3, (48)]\]. □

Let us consider \( G \) and \( H \). The functor \( \overline{H} \) yielding a subset of \( G \) is defined by the term

(Def. 19) the carrier of \( H \).

Now we state the propositions:

(120) If \( g_1, g_2 \in \overline{H} \), then \( g_1 + g_2 \in \overline{H} \). The theorem is a consequence of (96).

(121) If \( g \in \overline{H} \), then \( -g \in \overline{H} \). The theorem is a consequence of (97).

(122) \( \overline{H} + \overline{H} = \overline{H} \). The theorem is a consequence of (121), (120), and (68).

(123) \( -\overline{H} = \overline{H} \). The theorem is a consequence of (121) and (69).

(124) (i) if \( \overline{H}_1 + \overline{H}_2 = \overline{H}_2 + \overline{H}_1 \), then there exists a strict subgroup \( H \) of \( G \) such that the carrier of \( H = \overline{H}_1 + \overline{H}_2 \), and

(ii) if there exists \( H \) such that the carrier of \( H = \overline{H}_1 + \overline{H}_2 \), then \( \overline{H}_1 + \overline{H}_2 = \overline{H}_2 + \overline{H}_1 \).

The theorem is a consequence of (121), (16), (120), (55), and (98).

(125) Suppose \( G \) is an Abelian additive group. Then there exists a strict subgroup \( H \) of \( G \) such that the carrier of \( H = \overline{H}_1 + \overline{H}_2 \). The theorem is a consequence of (71) and (124).

Let us consider \( G, H_1 \), and \( H_2 \). The functor \( H_1 \cap H_2 \) yielding a strict subgroup of \( G \) is defined by

(Def. 20) the carrier of \( it = \overline{H}_1 \cap \overline{H}_2 \).

Now we state the propositions:

(126) (i) for every subgroup \( H \) of \( G \) such that \( H = H_1 \cap H_2 \) holds the carrier of \( H \) = (the carrier of \( H_1 \)) \cap (the carrier of \( H_2 \)), and

(ii) for every strict subgroup \( H \) of \( G \) such that the carrier of \( H = (the \ carrier \ of \ H_1) \cap (the \ carrier \ of \ H_2) \) holds \( H = H_1 \cap H_2 \).

(127) \( \overline{H}_1 \cap \overline{H}_2 = \overline{H}_1 \cap \overline{H}_2 \).

(128) \( x \in H_1 \cap H_2 \) if and only if \( x \in H_1 \) and \( x \in H_2 \).

(129) Let us consider a strict subgroup \( H \) of \( G \). Then \( H \cap H = H \). The theorem is a consequence of (105).

Let us consider \( G, H_1 \), and \( H_2 \). Note that the functor \( H_1 \cap H_2 \) is commutative.
(130) \((H_1 \cap H_2) \cap H_3 = H_1 \cap (H_2 \cap H_3)\). The theorem is a consequence of (105).

(131) (i) \(0_G \cap H = 0_G\), and

(ii) \(H \cap 0_G = 0_G\).

The theorem is a consequence of (111).

(132) Let us consider a strict additive group \(G\), and a strict subgroup \(H\) of \(G\). Then

(i) \(H \cap \Omega_G = H\), and

(ii) \(\Omega_G \cap H = H\).

(133) Let us consider a strict additive group \(G\). Then \(\Omega_G \cap \Omega_G = G\).

(134) \(H_1 \cap H_2\) is subgroup of \(H_1\) and subgroup of \(H_2\).

(135) Let us consider a subgroup \(H_1\) of \(G\). Then \(H_1\) is a subgroup of \(H_2\) if and only if the additive magma of \(H_1 \cap H_2 = \text{the additive magma of } H_1\).

(136) If \(H_1\) is a subgroup of \(H_2\), then \(H_1 \cap H_3\) is a subgroup of \(H_2\). The theorem is a consequence of (102).

(137) If \(H_1\) is subgroup of \(H_2\) and subgroup of \(H_3\), then \(H_1\) is a subgroup of \(H_2 \cap H_3\). The theorem is a consequence of (86), (128), and (104).

(138) If \(H_1\) is a subgroup of \(H_2\), then \(H_1 \cap H_3\) is a subgroup of \(H_2 \cap H_3\). The theorem is a consequence of (126) and (103).

(139) If \(H_1\) is finite or \(H_2\) is finite, then \(H_1 \cap H_2\) is finite.

Let us consider \(G\), \(H\), and \(A\). The functors: \(A + H\) and \(H + A\) yielding subsets of \(G\) are defined by terms,

(Def. 21) \(A + \overline{H}\),

(Def. 22) \(\overline{H} + A\),

respectively. Now we state the propositions:

(140) \(x \in A + H\) if and only if there exists \(g_1\) and there exists \(g_2\) such that \(x = g_1 + g_2\) and \(g_1 \in A\) and \(g_2 \in H\).

(141) \(x \in H + A\) if and only if there exists \(g_1\) and there exists \(g_2\) such that \(x = g_1 + g_2\) and \(g_1 \in H\) and \(g_2 \in A\).

(142) \((A + B) + H = A + (B + H)\).

(143) \((A + H) + B = A + (H + B)\).

(144) \((H + A) + B = H + (A + B)\).

(145) \((A + H_1) + H_2 = A + (H_1 + H_2)\).

(146) \((H_1 + A) + H_2 = H_1 + (A + H_2)\).

(147) \((H_1 + \overline{H_2}) + A = H_1 + (\overline{H_2} + A)\).

(148) If \(G\) is an Abelian additive group, then \(A + H = H + A\).
Let us consider $G$, $H$, and $a$. The functors: $a + H$ and $H + a$ yielding subsets of $G$ are defined by terms,

(Def. 23) $a + H$

(Def. 24) $H + a$

respectively. Now we state the propositions:

(149) $x \in a + H$ if and only if there exists $g$ such that $x = a + g$ and $g \in H$.
    The theorem is a consequence of (73).

(150) $x \in H + a$ if and only if there exists $g$ such that $x = g + a$ and $g \in H$.
    The theorem is a consequence of (74).

(151) $(a + b) + H = a + (b + H)$.

(152) $(a + H) + b = a + (H + b)$.

(153) $(H + a) + b = H + (a + b)$.

(154) (i) $a \in a + H$, and
    (ii) $a \in H + a$.
    The theorem is a consequence of (92), (149), and (150).

(155) (i) $0_G + H = H$
    (ii) $H + 0_G = H$.

(156) (i) $0_G + a = \{a\}$, and
    (ii) $a + 0_G = \{a\}$.
    The theorem is a consequence of (64).

(157) (i) $a + \Omega_G = \text{the carrier of } G$, and
    (ii) $\Omega_G + a = \text{the carrier of } G$.
    The theorem is a consequence of (63).

(158) If $G$ is an Abelian additive group, then $a + H = H + a$.

(159) $a \in H$ if and only if $a + H = H$. The theorem is a consequence of (149), (96), (97), and (92).

(160) $a + H = b + H$ if and only if $-b + a \in H$. The theorem is a consequence of (78), (83), and (159).

(161) $a + H = b + H$ if and only if $a + H$ meets $b + H$. The theorem is a consequence of (154), (149), (97), (13), (12), (96), and (160).

(162) $(a + b) + H \subseteq a + H + (b + H)$. The theorem is a consequence of (149) and (92).

(163) (i) $H \subseteq a + H + (-a + H)$, and
    (ii) $H \subseteq a + H + (a + H)$.
    The theorem is a consequence of (83) and (162).
(164) $2 \cdot a + H \subseteq a + H + (a + H)$. The theorem is a consequence of (26) and (162).

(165) $a \in H$ if and only if $H + a = \overline{H}$. The theorem is a consequence of (150), (96), (97), and (92).

(166) $H + a = H + b$ if and only if $b + -a \in H$. The theorem is a consequence of (83), (80), and (165).

(167) $H + a = H + b$ if and only if $H + a$ meets $H + b$. The theorem is a consequence of (154), (150), (97), (12), (13), (96), and (166).

(168) $(H + a) + b \subseteq H + a + (H + b)$. The theorem is a consequence of (92), (150), and (80).

(169) (i) $\overline{H} \subseteq H + a + (H + -a)$, and

(ii) $\overline{H} \subseteq H + -a + (H + a)$.

The theorem is a consequence of (80), (83), and (168).

(170) $H + 2 \cdot a \subseteq H + a + (H + a)$. The theorem is a consequence of (80), (26), and (168).

(171) $a + H_1 \cap H_2 = (a + H_1) \cap (a + H_2)$. The theorem is a consequence of (149), (128), and (6).

(172) $H_1 \cap H_2 + a = (H_1 + a) \cap (H_2 + a)$. The theorem is a consequence of (150), (128), and (6).

(173) There exists a strict subgroup $H_1$ of $G$ such that the carrier of $H_1 = a + H_2 + -a$. The theorem is a consequence of (154), (74), (149), (97), (150), (16), (73), (56), (96), and (98).

(174) $a + H \approx b + H$.

Proof: Define $\mathcal{P}[\text{object, object}] \equiv$ there exists $g_1$ such that $g_1 = g_1$ and $g_2 = b + -a + g_1$. For every object $x$ such that $x \in a + H$ there exists an object $y$ such that $\mathcal{P}[x, y]$. Consider $f$ being a function such that dom $f = a + H$ and for every object $x$ such that $x \in a + H$ holds $\mathcal{P}[x, f(x)]$ from [4, Sch. 1]. rng $f = b + H$. $f$ is one-to-one. □

(175) $a + H \approx b + H$.

Proof: Define $\mathcal{P}[\text{object, object}] \equiv$ there exists $g_1$ such that $g_1 = g_1$ and $g_2 = -a + g_1 + b$. For every object $x$ such that $x \in a + H$ there exists an object $y$ such that $\mathcal{P}[x, y]$. Consider $f$ being a function such that dom $f = a + H$ and for every object $x$ such that $x \in a + H$ holds $\mathcal{P}[x, f(x)]$ from [4, Sch. 1]. rng $f = H + b$. $f$ is one-to-one. □

(176) $H + a \approx H + b$. The theorem is a consequence of (175).

(177) (i) $\overline{H} \approx a + H$, and

(ii) $\overline{H} \approx H + a$.

The theorem is a consequence of (83), (174), and (176).
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(178) \begin{align*}
(i) \quad H &= \overline{a + H}, \text{ and} \\
(ii) \quad \overline{H} &= \overline{H + a}.
\end{align*}

(179) Let us consider a finite subgroup $H$ of $G$. Then there exist finite sets $B$, $C$ such that

(i) $B = a + H$, and

(ii) $C = H + a$, and

(iii) $\overline{H} = \overline{B}$, and

(iv) $\overline{H} = \overline{C}$.

The theorem is a consequence of (177).

Let us consider $G$ and $H$. The functors: the left cosets of $H$ and the right cosets of $H$ yielding families of subsets of $G$ are defined by conditions,

(Def. 25) $A \in$ the left cosets of $H$ iff there exists $a$ such that $A = a + H$,

(Def. 26) $A \in$ the right cosets of $H$ iff there exists $a$ such that $A = H + a$, respectively. Now we state the propositions:

(180) If $G$ is finite, then the right cosets of $H$ is finite and the left cosets of $H$ is finite.

(181) (i) $\overline{H} \in$ the left cosets of $H$, and

(ii) $\overline{H} \in$ the right cosets of $H$.

The theorem is a consequence of (83).

(182) The left cosets of $H \approx$ the right cosets of $H$.

Proof: Define $\mathcal{P}[\text{object, object}] \equiv$ there exists $g$ such that $g_1 = g + H$ and $g_2 = H + -g$. For every object $x$ such that $x \in$ the left cosets of $H$ there exists an object $y$ such that $\mathcal{P}[x, y]$. Consider $f$ being a function such that $\text{dom } f = \text{the left cosets of } H$ and for every object $x$ such that $x \in$ the left cosets of $H$ holds $\mathcal{P}[x, f(x)]$ from $[4]$ Sch. 1]. $\text{rng } f = \text{the right cosets of } H$. $f$ is one-to-one. $\Box$

(183) (i) $\bigcup(\text{the left cosets of } H) = \text{the carrier of } G$, and

(ii) $\bigcup(\text{the right cosets of } H) = \text{the carrier of } G$.

The theorem is a consequence of (87), (149), and (150).

(184) The left cosets of $0_G = \text{the set of all } \{a\}$. The theorem is a consequence of (156).

(185) The right cosets of $0_G = \text{the set of all } \{a\}$. The theorem is a consequence of (156).

Let us consider a strict subgroup $H$ of $G$. Now we state the propositions:

(186) If the left cosets of $H = \text{the set of all } \{a\}$, then $H = 0_G$. The theorem is a consequence of (87), (149), (92), and (6).
(187) If the right cosets of $H = \{a\}$, then $H = 0_G$. The theorem is a consequence of (87), (150), (92), and (6).

(188) (i) the left cosets of $\Omega_G = \{\text{the carrier of } G\}$, and
(ii) the right cosets of $\Omega_G = \{\text{the carrier of } G\}$.

The theorem is a consequence of (157).

Let us consider a strict additive group $G$ and a strict subgroup $H$ of $G$. Now we state the propositions:

(189) If the left cosets of $H = \{\text{the carrier of } G\}$, then $H = G$. The theorem is a consequence of (149), (6), and (108).

(190) If the right cosets of $H = \{\text{the carrier of } G\}$, then $H = G$. The theorem is a consequence of (150), (6), and (108).

Let us consider $G$ and $H$. The functor $|\bullet : H|$ yielding a cardinal number is defined by the term

(Def. 27) $\overline{\alpha}$, where $\alpha$ is the left cosets of $H$.

Now we state the proposition:

(191) (i) $|\bullet : H| = \overline{\alpha}$, and
(ii) $|\bullet : H| = \overline{\beta}$,

where $\alpha$ is the left cosets of $H$ and $\beta$ is the right cosets of $H$.

Let us consider $G$ and $H$. Assume the left cosets of $H$ is finite. The functor $|\bullet : H|_N$ yielding an element of $N$ is defined by

(Def. 28) there exists a finite set $B$ such that $B = \{a\}$, and $it = \overline{B}$.

Now we state the proposition:

(192) Suppose the left cosets of $H$ is finite. Then

(i) there exists a finite set $B$ such that $B = \{a\}$ and $|\bullet : H|_N = \overline{B}$, and

(ii) there exists a finite set $C$ such that $C = \{a\}$ and $|\bullet : H|_N = \overline{C}$.

The theorem is a consequence of (182).

Let us consider a finite additive group $G$ and a subgroup $H$ of $G$. Now we state the propositions:

(193) **LAGRANGE THEOREM FOR ADDITIVE GROUPS:**

$\overline{G} = \overline{H} \cdot |\bullet : H|_N$. The theorem is a consequence of (179), (174), (161), and (183).

(194) $\overline{H} \mid \overline{G}$. The theorem is a consequence of (193).
Groups – additive notation

Let us consider a finite additive group $G$, subgroups $I$, $H$ of $G$, and a subgroup $J$ of $H$. Suppose $I = J$. Then $|\cdot : I|_N = |\cdot : J|_N \cdot |\cdot : H|_N$. The theorem is a consequence of (193).

Let us consider a strict additive group $G$, and a strict subgroup $H$ of $G$. Suppose the left cosets of $H$ is finite and $|\cdot : H|_N = 1$. Then $H = G$. The theorem is a consequence of (183) and (189).

$|\cdot : 0_G|_N = \overline{G}$.

Proof: Define $\mathcal{F}(\text{object}) = \{S_1\}$. Consider $f$ being a function such that $\text{dom } f = \text{the carrier of } G$ and for every object $x$ such that $x \in \text{the carrier of } G$ holds $f(x) = \mathcal{F}(x)$ from [14, Sch. 3]. $\text{rng } f = \text{the left cosets of } 0_G$. $f$ is one-to-one by [17, (3)]. □

Let us consider a finite additive group $G$. Then $|\cdot : 0_G|_N = \overline{G}$. The theorem is a consequence of (193) and (115).

Let us consider a finite additive group $G$, and a strict subgroup $H$ of $G$. Suppose $|\cdot : H|_N = \overline{G}$. Then $H = 0_G$. The theorem is a consequence of (193) and (116).

Let us consider a strict subgroup $H$ of $G$. Suppose the left cosets of $H$ is finite and $|\cdot : H| = \overline{G}$. Then

(i) $G$ is finite, and

(ii) $H = 0_G$.

The theorem is a consequence of (200).

3. Classes of Conjugation and Normal Subgroups – GROUP_3

From now on $x, y, y_1, y_2$ denote sets, $G$ denotes an additive group, $a, b, c, d, g, h$ denote elements of $G$, $A, B, C, D$ denote subsets of $G$, $H, H_1, H_2, H_3$ denote subgroups of $G$, $n$ denotes a natural number, and $i$ denotes an integer.

Now we state the propositions:

(i) $a + b + -b = a$, and

(ii) $a + -b + b = a$, and

(iii) $-b + b + a = a$, and

(iv) $b + -b + a = a$, and

(v) $a + (b + -b) = a$, and

(vi) $a + (-b + b) = a$, and

(vii) $-b + (b + a) = a$, and
(viii) $b + (-b + a) = a$.

(203) $G$ is an Abelian additive group if and only if the addition of $G$ is commutative.

(204) $0_G$ is Abelian.

(205) If $A \subseteq B$ and $C \subseteq D$, then $A + C \subseteq B + D$.

(206) If $A \subseteq B$, then $a + A \subseteq a + B$ and $A + a \subseteq B + a$.

(207) If $H_1$ is a subgroup of $H_2$, then $a + H_1 \subseteq a + H_2$ and $H_1 + a \subseteq H_2 + a$. The theorem is a consequence of (205).

(208) $a + H = \{a\} + H$.

(209) $H + a = H + \{a\}$.

(210) $(A + a) + H = A + (a + H)$. The theorem is a consequence of (142).

(211) $(a + H) + A = a + (H + A)$. The theorem is a consequence of (143).

(212) $(A + H) + a = A + (H + a)$. The theorem is a consequence of (143).

(213) $(H + a) + A = H + (a + A)$. The theorem is a consequence of (144).

(214) $(H_1 + a) + H_2 = H_1 + (a + H_2)$.

Let us consider $G$. The functor SubGr $G$ yielding a set is defined by

(Def. 29) for every object $x$, $x \in it$ iff $x$ is a strict subgroup of $G$.

Note that SubGr $G$ is non empty.

Now we state the propositions:

(215) Let us consider a strict additive group $G$. Then $G \in \text{SubGr} G$. The theorem is a consequence of (100).

(216) If $G$ is finite, then SubGr $G$ is finite.

PROOF: Define $\mathcal{P}[\text{object, object}] \equiv$ there exists a strict subgroup $H$ of $G$ such that $S_1 = H$ and $S_2 =$ the carrier of $H$. For every object $x$ such that $x \in \text{SubGr} G$ there exists an object $y$ such that $\mathcal{P}[x, y]$. Consider $f$ being a function such that $\text{dom} f = \text{SubGr} G$ and for every object $x$ such that $x \in \text{SubGr} G$ holds $\mathcal{P}[x, f(x)]$ from [4, Sch. 1]. $\text{rng} f \subseteq 2^\alpha$, where $\alpha$ is the carrier of $G$. $f$ is one-to-one. □

Let us consider $G$, $a$, and $b$. The functor $a \cdot b$ yielding an element of $G$ is defined by the term

(Def. 30) $-b + a + b$.

Now we state the propositions:

(217) If $a \cdot g = b \cdot g$, then $a = b$. The theorem is a consequence of (6).

(218) $0_G \cdot a = 0_G$.

(219) If $a \cdot b = 0_G$, then $a = 0_G$. The theorem is a consequence of (11) and (7).

(220) $a \cdot 0_G = a$. The theorem is a consequence of (8).
Groups – additive notation

(221) $a \cdot a = a$.

(222) (i) $a \cdot (-a) = a$, and

(ii) $(-a) \cdot a = -a$.

(223) $a \cdot b = a$ if and only if $a + b = b + a$. The theorem is a consequence of (12).

(224) $(a + b) \cdot g = a \cdot g + b \cdot g$.

(225) $a \cdot g \cdot h = a \cdot (g + h)$. The theorem is a consequence of (16).

(226) (i) $a \cdot b \cdot (-b) = a$, and

(ii) $a \cdot (-b) \cdot b = a$.

The theorem is a consequence of (225) and (220).

(227) $(-a) \cdot b = -a \cdot b$. The theorem is a consequence of (16).

(228) $(n \cdot a) \cdot b = n \cdot (a \cdot b)$.

(229) $(i \cdot a) \cdot b = i \cdot (a \cdot b)$. The theorem is a consequence of (29) and (227).

(230) If $G$ is an Abelian additive group, then $a \cdot b = a$. The theorem is a consequence of (202).

(231) If for every $a$ and $b$, $a \cdot b = a$, then $G$ is Abelian. The theorem is a consequence of (223).

Let us consider $G$, $A$, and $B$. The functor $A \cdot B$ yielding a subset of $G$ is defined by the term

(Def. 31) $\{g \cdot h : g \in A \text{ and } h \in B\}$.

Now we state the propositions:

(232) $x \in A \cdot B$ if and only if there exists $g$ and there exists $h$ such that $x = g \cdot h$ and $g \in A$ and $h \in B$.

(233) $A \cdot B \neq \emptyset$ if and only if $A \neq \emptyset$ and $B \neq \emptyset$. The theorem is a consequence of (232).

(234) $A \cdot B \subseteq -B + A + B$.

(235) $(A + B) \cdot C \subseteq A \cdot C + B \cdot C$. The theorem is a consequence of (224).

(236) $A \cdot B \cdot C = A \cdot (B + C)$. The theorem is a consequence of (225).

(237) $(-A) \cdot B = -A \cdot B$. The theorem is a consequence of (227).

(238) $\{a\} \cdot \{b\} = \{a \cdot b\}$. The theorem is a consequence of (49), (64), (233), and (234).

(239) $\{a\} \cdot \{b, c\} = \{a \cdot b, a \cdot c\}$.

(240) $\{a, b\} \cdot \{c\} = \{a \cdot c, b \cdot c\}$.

(241) $\{a, b\} \cdot \{c, d\} = \{a \cdot c, a \cdot d, b \cdot c, b \cdot d\}$.

Let us consider $G$, $A$, and $g$. The functors: $A \cdot g$ and $g \cdot A$ yielding subsets of $G$ are defined by terms,
(Def. 32) \( A \cdot \{g\} \),
(Def. 33) \( \{g\} \cdot A \),
respectively. Now we state the propositions:

(242) \( x \in A \cdot g \) if and only if there exists \( h \) such that \( x = h \cdot g \) and \( h \in A \).
(243) \( x \in g \cdot A \) if and only if there exists \( h \) such that \( x = g \cdot h \) and \( h \in A \).
(244) \( g \cdot A \subseteq -A + g + A \). The theorem is a consequence of (243) and (74).
(245) \( A \cdot B \cdot g = A \cdot (B + g) \).
(246) \( A \cdot g \cdot B = A \cdot (g + B) \).
(247) \( g \cdot A \cdot B = g \cdot (A + B) \).
(248) \( A \cdot a \cdot b = A \cdot (a + b) \). The theorem is a consequence of (236) and (64).
(249) \( a \cdot A \cdot b = a \cdot (A + b) \).
(250) \( a \cdot b \cdot A = a \cdot (b + A) \). The theorem is a consequence of (238) and (236).
(251) \( A \cdot g = -g + A + g \). The theorem is a consequence of (234), (49), (74), (73), and (242).
(252) \( (A + B) \cdot a \subseteq A \cdot a + B \cdot a \).
(253) \( A \cdot 0_G = A \). The theorem is a consequence of (251), (83), and (8).
(254) If \( A \neq \emptyset \), then \( 0_G \cdot A = \{0_G\} \). The theorem is a consequence of (243) and (218).
(255) (i) \( A \cdot a \cdot (-a) = A \), and
(ii) \( A \cdot (-a) \cdot a = A \).
The theorem is a consequence of (248) and (253).
(256) \( G \) is an Abelian additive group if and only if for every \( A \) and \( B \) such that \( B \neq \emptyset \) holds \( A \cdot B = A \). The theorem is a consequence of (230), (238), and (231).
(257) \( G \) is an Abelian additive group if and only if for every \( A \) and \( g \), \( A \cdot g = A \).
The theorem is a consequence of (256), (238), and (231).
(258) \( G \) is an Abelian additive group if and only if for every \( A \) and \( g \) such that \( A \neq \emptyset \) holds \( g \cdot A = \{g\} \). The theorem is a consequence of (256), (238), and (231).

Let us consider \( G \), \( H \), and \( a \). The functor \( H \cdot a \) yielding a strict subgroup of \( G \) is defined by

(Def. 34) the carrier of \( it = H \cdot a \).

Now we state the propositions:

(259) \( x \in H \cdot a \) if and only if there exists \( g \) such that \( x = g \cdot a \) and \( g \in H \). The theorem is a consequence of (242).
(260) The carrier of \(H \cdot a = -a + H + a\). The theorem is a consequence of (251).

(261) \(H \cdot a \cdot b = H \cdot (a + b)\). The theorem is a consequence of (248) and (105).

Let us consider a strict subgroup \(H\) of \(G\). Now we state the propositions:

(262) \(H \cdot 0_G = H\). The theorem is a consequence of (253) and (105).

(263) (i) \(H \cdot a \cdot (-a) = H\), and

(ii) \(H \cdot (-a) \cdot a = H\).

The theorem is a consequence of (261) and (262).

Now we state the propositions:

(264) \((H_1 \cap H_2) \cdot a = H_1 \cdot a \cap (H_2 \cdot a)\). The theorem is a consequence of (259), (128), and (217).

(265) \(\overline{H} = H \cdot a\).

Proof: Define \(F\) (element of \(G\)) = \(\mathbb{1} \cdot a\). Consider \(f\) being a function from the carrier of \(G\) into the carrier of \(G\) such that for every \(g\), \(f(g) = F(g)\) from [15, Sch. 4]. Set \(g = f\) (the carrier of \(H\)). \(\text{rng} g = \text{the carrier of } H \cdot a\) by [46 (62)], (88), (242), [14 (47)]. \(g\) is one-to-one by [46, (62)], (88), [14, (47)], (217). \(\square\)

(266) \(H\) is finite if and only if \(H \cdot a\) is finite. The theorem is a consequence of (265).

Let us consider \(G\) and \(a\). Let \(H\) be a finite subgroup of \(G\). Observe that \(H \cdot a\) is finite.

Now we state the propositions:

(267) Let us consider a finite subgroup \(H\) of \(G\). Then \(\overline{H} = H \cdot a\).

(268) \(0_G \cdot a = 0_G\). The theorem is a consequence of (238) and (218).

(269) Let us consider a strict subgroup \(H\) of \(G\). If \(H \cdot a = 0_G\), then \(H = 0_G\).

The theorem is a consequence of (266), (115), (265), and (116).

(270) Let us consider an additive group \(G\), and an element \(a\) of \(G\). Then \(\Omega_G \cdot a = \Omega_G\). The theorem is a consequence of (225), (220), and (259).

(271) Let us consider a strict subgroup \(H\) of \(G\). If \(H \cdot a = G\), then \(H = G\).

The theorem is a consequence of (259), (217), and (108).

(272) \(|\bullet : H| = |\bullet : H \cdot a|\).

Proof: Define \(P\) (object, object) = there exists \(b\) such that \(\mathbb{1} = b + H\) and \(\mathbb{2} = b \cdot a + H \cdot a\). For every object \(x\) such that \(x \in \text{the left cosets of } H\) there exists an object \(y\) such that \(P[x, y]\). Consider \(f\) being a function such that \(\text{dom } f = \text{the left cosets of } H\) and for every object \(x\) such that \(x \in \text{the left cosets of } H\) holds \(P[x, f(x)]\) from [4 Sch. 1]. For every \(x, y_1,\) and \(y_2\) such that \(x \in \text{the left cosets of } H\) and \(P[x, y_1]\) and \(P[x, y_2]\) holds \(y_1 = y_2\). \(\text{rng } f = \text{the left cosets of } H \cdot a\). \(f\) is one-to-one. \(\square\)
(273) If the left cosets of $H$ is finite, then $|\bullet : H|_N = |\bullet : H \cdot a|_N$. The theorem is a consequence of (272).

(274) If $G$ is an Abelian additive group, then for every strict subgroup $H$ of $G$ and for every $a$, $H \cdot a = H$. The theorem is a consequence of (260), (158), (153), (155), and (105).

Let us consider $G$, $a$, and $b$. We say that $a$ and $b$ are conjugated if and only if

(Def. 35) there exists $g$ such that $a = b \cdot g$.

Now we state the proposition:

(275) $a$ and $b$ are conjugated if and only if there exists $g$ such that $b = a \cdot g$.

The theorem is a consequence of (226).

Let us consider $G$, $a$, and $b$. Observe that $a$ and $b$ are conjugated is reflexive and symmetric.

Now we state the propositions:

(276) If $a$ and $b$ are conjugated and $b$ and $c$ are conjugated, then $a$ and $c$ are conjugated. The theorem is a consequence of (225).

(277) If $a$ and $0_G$ are conjugated or $0_G$ and $a$ are conjugated, then $a = 0_G$.

The theorem is a consequence of (275) and (219).

(278) $a \cdot \Omega_G = \{b : a$ and $b$ are conjugated\}. The theorem is a consequence of (243).

Let us consider $G$ and $a$. The functor $a^\bullet$ yielding a subset of $G$ is defined by the term

(Def. 36) $a \cdot \Omega_G$.

Now we state the propositions:

(279) $x \in a^\bullet$ if and only if there exists $b$ such that $b = x$ and $a$ and $b$ are conjugated. The theorem is a consequence of (278).

(280) $a \in b^\bullet$ if and only if $a$ and $b$ are conjugated. The theorem is a consequence of (279).

(281) $a \cdot g \in a^\bullet$.

(282) $a \in a^\bullet$.

(283) If $a \in b^\bullet$, then $b \in a^\bullet$. The theorem is a consequence of (280).

(284) $a^\bullet = b^\bullet$ if and only if $a^\bullet$ meets $b^\bullet$. The theorem is a consequence of (280), (279), and (276).

(285) $a^\bullet = \{0_G\}$ if and only if $a = 0_G$. The theorem is a consequence of (280), (279), and (277).

(286) $a^\bullet + A = A + a^\bullet$. The theorem is a consequence of (280), (202), (226), (224), (221), (225), (279), and (275).
Let us consider $G$, $A$, and $B$. We say that $A$ and $B$ are conjugated if and only if

(Def. 37) there exists $g$ such that $A = B \cdot g$.

Now we state the propositions:

(287) $A$ and $B$ are conjugated if and only if there exists $g$ such that $B = A \cdot g$.

The theorem is a consequence of (255).

(288) $A$ and $A$ are conjugated. The theorem is a consequence of (253).

(289) If $A$ and $B$ are conjugated, then $B$ and $A$ are conjugated. The theorem is a consequence of (255).

Let us consider $G$, $A$, and $B$. Let us observe that $A$ and $B$ are conjugated is reflexive and symmetric.

Now we state the propositions:

(290) If $A$ and $B$ are conjugated and $B$ and $C$ are conjugated, then $A$ and $C$ are conjugated. The theorem is a consequence of (248).

(291) \{$a\$} and \{$b\$} are conjugated if and only if $a$ and $b$ are conjugated.

PROOF: If \{$a\$} and \{$b\$} are conjugated, then $a$ and $b$ are conjugated by (287), (238), (275), \([17\ (3)]\). Consider $g$ such that $a \cdot g = b$. \{$b\$} = \{$a\$} \cdot g$. □

(292) If $A$ and $\overline{H_1}$ are conjugated, then there exists a strict subgroup $H_2$ of $G$ such that the carrier of $H_2 = A$.

Let us consider $G$ and $A$. The functor $A^\bullet$ yielding a family of subsets of $G$ is defined by the term

(Def. 38) \{$B : A$ and $B$ are conjugated\}.

Now we state the propositions:

(293) $x \in A^\bullet$ if and only if there exists $B$ such that $x = B$ and $A$ and $B$ are conjugated.

(294) $A \in B^\bullet$ if and only if $A$ and $B$ are conjugated.

(295) $A \cdot g \in A^\bullet$. The theorem is a consequence of (287).

(296) $A \in A^\bullet$.

(297) If $A \in B^\bullet$, then $B \in A^\bullet$. The theorem is a consequence of (294).

(298) $A^\bullet = B^\bullet$ if and only if $A^\bullet$ meets $B^\bullet$. The theorem is a consequence of (294) and (290).

(299) \{$a\$}^\bullet = \{$b \in a^\bullet\$. The theorem is a consequence of (287), (275), (280), (238), and (291).

(300) If $G$ is finite, then $A^\bullet$ is finite.

Let us consider $G$, $H_1$, and $H_2$. We say that $H_1$ and $H_2$ are conjugated if and only if
(Def. 39) there exists \( g \) such that the additive magma of \( H_1 = H_2 \cdot g \).

Now we state the propositions:

(301) Let us consider strict subgroups \( H_1, H_2 \) of \( G \). Then \( H_1 \) and \( H_2 \) are conjugated if and only if there exists \( g \) such that \( H_2 = H_1 \cdot g \). The theorem is a consequence of (263).

(302) Let us consider a strict subgroup \( H_1 \) of \( G \). Then \( H_1 \) and \( H_1 \) are conjugated. The theorem is a consequence of (262).

(303) Let us consider strict subgroups \( H_1, H_2 \) of \( G \). If \( H_1 \) and \( H_2 \) are conjugated, then \( H_2 \) and \( H_1 \) are conjugated. The theorem is a consequence of (263).

Let us consider \( G \). Let \( H_1, H_2 \) be strict subgroups of \( G \). Observe that \( H_1 \) and \( H_2 \) are conjugated is reflexive and symmetric.

Now we state the proposition:

(304) Let us consider strict subgroups \( H_1, H_2 \) of \( G \). Suppose \( H_1 \) and \( H_2 \) are conjugated and \( H_2 \) and \( H_3 \) are conjugated. Then \( H_1 \) and \( H_3 \) are conjugated. The theorem is a consequence of (261).

In the sequel \( L \) denotes a subset of SubGr \( G \).

Let us consider \( G \) and \( H \). The functor \( H^\bullet \) yielding a subset of SubGr \( G \) is defined by

(Def. 40) for every object \( x, x \in it \) iff there exists a strict subgroup \( H_1 \) of \( G \) such that \( x = H_1 \) and \( H \) and \( H_1 \) are conjugated.

Now we state the propositions:

(305) If \( x \in H^\bullet \), then \( x \) is a strict subgroup of \( G \).

(306) Let us consider strict subgroups \( H_1, H_2 \) of \( G \). Then \( H_1 \in H^\bullet \) if and only if \( H_1 \) and \( H_2 \) are conjugated.

Let us consider a strict subgroup \( H \) of \( G \). Now we state the propositions:

(307) \( H \cdot g \in H^\bullet \). The theorem is a consequence of (301).

(308) \( H \in H^\bullet \).

Let us consider strict subgroups \( H_1, H_2 \) of \( G \). Now we state the propositions:

(309) If \( H_1 \in H^\bullet \), then \( H_2 \in H^\bullet \). The theorem is a consequence of (306).

(310) \( H_1^\bullet = H_2^\bullet \) if and only if \( H_1^\bullet \) meets \( H_2^\bullet \). The theorem is a consequence of (308), (305), (306), and (304).

Now we state the propositions:

(311) If \( G \) is finite, then \( H^\bullet \) is finite.

(312) Let us consider a strict subgroup \( H_1 \) of \( G \). Then \( H_1 \) and \( H_2 \) are conjugated if and only if \( \overline{H_1} \) and \( \overline{H_2} \) are conjugated.
Let us consider \(G\). Let \(I_1\) be a subgroup of \(G\). We say that \(I_1\) is normal if and only if

(Def. 41) for every \(a\), \(I_1 \cdot a = \text{the additive magma of } I_1\).

Let us note that there exists a subgroup of \(G\) which is strict and normal.
From now on \(N_2\) denotes a normal subgroup of \(G\).

Now we state the propositions:

(313) (i) \(0_G\) is normal, and
     (ii) \(\Omega_G\) is normal.

(314) Let us consider strict, normal subgroups \(N_1, N_2\) of \(G\). Then \(N_1 \cap N_2\) is normal. The theorem is a consequence of (264).

(315) Let us consider a strict subgroup \(H\) of \(G\). If \(G\) is an Abelian additive group, then \(H\) is normal.

(316) \(H\) is a normal subgroup of \(G\) if and only if for every \(a\), \(a + H = H + a\).
     The theorem is a consequence of (260), (79), (151), (83), (153), (155), and (105).

Let us consider a subgroup \(H\) of \(G\). Now we state the propositions:

(317) \(H\) is a normal subgroup of \(G\) if and only if for every \(a\), \(a + H \subseteq H + a\).
     The theorem is a consequence of (316), (205), (151), (155), (152), (80), and (83).

(318) \(H\) is a normal subgroup of \(G\) if and only if for every \(a\), \(H + a \subseteq a + H\).
     The theorem is a consequence of (316), (205), (151), (155), (152), (80), and (83).

(319) \(H\) is a normal subgroup of \(G\) if and only if for every \(A\), \(A + H = H + A\).
     The theorem is a consequence of (140), (149), (316), (150), and (141).

Let us consider a strict subgroup \(H\) of \(G\). Now we state the propositions:

(320) \(H\) is a normal subgroup of \(G\) if and only if for every \(a\), \(H\) is a subgroup of \(H \cdot a\).
     The theorem is a consequence of (100), (260), (80), (83), (207), and (318).

(321) \(H\) is a normal subgroup of \(G\) if and only if for every \(a\), \(H \cdot a\) is a subgroup of \(H\).
     The theorem is a consequence of (100), (260), (80), (83), (207), and (317).

(322) \(H\) is a normal subgroup of \(G\) if and only if \(H^\bullet = \{H\}\).
     PROOF: If \(H\) is a normal subgroup of \(G\), then \(H^\bullet = \{H\}\) by (301), (308), [17] (31)]. \(H\) is normal. \(\Box\)

(323) \(H\) is a normal subgroup of \(G\) if and only if for every \(a\) such that \(a \in H\) holds \(a^\bullet \subseteq \overline{H}\).
     The theorem is a consequence of (279), (275), (259), and (226).
Let us consider strict, normal subgroups $N_1, N_2$ of $G$. Now we state the propositions:

(324) $N_1 + N_2 = N_2 + N_1$.

(325) There exists a strict, normal subgroup $N$ of $G$ such that the carrier of $N = N_1 + N_2$. The theorem is a consequence of (124), (75), (316), (76), and (77).

Now we state the propositions:

(326) Let us consider a normal subgroup $N$ of $G$. Then the left cosets of $N$ = the right cosets of $N$. The theorem is a consequence of (316).

(327) Let us consider a subgroup $H$ of $G$. Suppose the left cosets of $H$ is finite and $|\cdot : H|_N = 2$. Then $H$ is a normal subgroup of $G$.

**Proof:** Define $P[\text{object}, \text{object}] \equiv \text{there exists } a \text{ such that } S_1 = A \cdot a$ and $S_2 = N(A) + a$. For every object $x$ such that $x \in A^\bullet$ there exists an object $y$ such that $P[x, y]$. Consider $f$ being a function such that $\text{dom } f = A^\bullet$ and for every object $x$ such that $x \in A^\bullet$ holds $P[x, f(x)]$ from [4, Sch. 1]. For every $x, y_1, y_2$ such that $x \in A^\bullet$ and $P[x, y_1]$ and $P[x, y_2]$ holds $y_1 = y_2$. $\text{rng } f$ = the right cosets of $N(A)$. $f$ is one-to-one. □

(330) Suppose $A^\bullet$ is finite or the left cosets of $N(A)$ is finite. Then there exists a finite set $C$ such that

(i) $C = A^\bullet$, and

(ii) $|\cdot : N(A)|_N$. 

Let us consider $G$ and $A$. The functor $N(A)$ yielding a strict subgroup of $G$ is defined by

(Def. 42) the carrier of it = $\{h : A \cdot h = A\}$.

Now we state the propositions:

(328) $x \in N(A)$ if and only if there exists $h$ such that $x = h$ and $A \cdot h = A$.

(329) $A^\bullet = |\cdot : N(A)|_N$.

**Proof:** Define $P[\text{object}, \text{object}] \equiv \text{there exists } a \text{ such that } S_1 = A \cdot a$ and $S_2 = N(A) + a$. For every object $x$ such that $x \in A^\bullet$ there exists an object $y$ such that $P[x, y]$. Consider $f$ being a function such that $\text{dom } f = A^\bullet$ and for every object $x$ such that $x \in A^\bullet$ holds $P[x, f(x)]$ from [4, Sch. 1]. For every $x, y_1, y_2$ such that $x \in A^\bullet$ and $P[x, y_1]$ and $P[x, y_2]$ holds $y_1 = y_2$. $\text{rng } f$ = the right cosets of $N(A)$. $f$ is one-to-one. □

(330) Suppose $A^\bullet$ is finite or the left cosets of $N(A)$ is finite. Then there exists a finite set $C$ such that

(i) $C = A^\bullet$, and

(ii) $|\cdot : N(A)|_N$. 

(324) $N_1 + N_2 = N_2 + N_1$. 

(325) There exists a strict, normal subgroup $N$ of $G$ such that the carrier of $N = N_1 + N_2$. The theorem is a consequence of (124), (75), (316), (76), and (77).
The theorem is a consequence of (329).

\[ (331) \quad \overline{a^*} = |\cdot : N(\{a\})|. \]

**Proof:** Define \( \mathcal{F}(\text{object}) = \{S_1\} \). Consider \( f \) being a function such that \( \text{dom } f = a^* \) and for every object \( x \) such that \( x \in a^* \) holds \( f(x) = \mathcal{F}(x) \) from \([14, \text{Sch. 3}]\). \( \text{rng } f = \{a\}^* \). \( f \) is one-to-one by \([17, (3)]\). \( \square \)

(332) Suppose \( a^* \) is finite or the left cosets of \( N(\{a\}) \) is finite. Then there exists a finite set \( C \) such that

(i) \( C = a^* \), and

(ii) \( \overline{C} = |\cdot : N(\{a\})|_N \).

The theorem is a consequence of (331).

Let us consider \( G \) and \( H \). The functor \( N(H) \) yielding a strict subgroup of \( G \) is defined by the term

(Def. 43) \( N(\overline{H}) \).

Let us consider a strict subgroup \( H \) of \( G \). Now we state the propositions:

(333) \( x \in N(H) \) if and only if there exists \( h \) such that \( x = h \) and \( H \cdot h = H \).

The theorem is a consequence of (328).

(334) \( \overline{H^*} = |\cdot : N(H)| \).

**Proof:** Define \( \mathcal{P}[\text{object,object}] \equiv \) there exists a strict subgroup \( H_1 \) of \( G \) such that \( S_1 = H_1 \) and \( S_2 = \overline{H_1} \). For every object \( x \) such that \( x \in H^* \) there exists an object \( y \) such that \( \mathcal{P}[x, y] \). Consider \( f \) being a function such that \( \text{dom } f = H^* \) and for every object \( x \) such that \( x \in H^* \) holds \( \mathcal{P}[x, f(x)] \) from \([4, \text{Sch. 1}]\). \( \text{rng } f = \overline{H^*} \). \( f \) is one-to-one. \( \square \)

(335) Suppose \( H^* \) is finite or the left cosets of \( N(H) \) is finite. Then there exists a finite set \( C \) such that

(i) \( C = H^* \), and

(ii) \( \overline{C} = |\cdot : N(H)|_N \).

The theorem is a consequence of (334).

Now we state the proposition:

(336) Let us consider a strict additive group \( G \), and a strict subgroup \( H \) of \( G \).

Then \( H \) is a normal subgroup of \( G \) if and only if \( N(H) = G \). The theorem is a consequence of (333) and (108).

Let us consider a strict additive group \( G \). Now we state the propositions:

(337) \( N(0_G) = G \). The theorem is a consequence of (313) and (336).

(338) \( N(\Omega_G) = G \). The theorem is a consequence of (313) and (336).
4. Topological Groups – \texttt{TOPGRP}_1

In the sequel, \( S, R \) denote 1-sorted structures, \( X \) denotes a subset of \( R \), \( T \) denotes a topological structure, \( x \) denotes a set, \( H \) denotes a non empty additive magma, \( P, Q, P_1, Q_1 \) denote subsets of \( H \), and \( h \) denotes an element of \( H \).

Now we state the proposition:

(339) If \( P \subseteq P_1 \) and \( Q \subseteq Q_1 \), then \( P + Q \subseteq P_1 + Q_1 \).

Let us assume that \( P \subseteq Q \). Now we state the propositions:

(340) \( P + h \subseteq Q + h \). The theorem is a consequence of (74).
(341) \( h + P \subseteq h + Q \). The theorem is a consequence of (73).

From now on \( a \) denotes an element of \( G \).

Now we state the propositions:

(342) \( a \in -A \) if and only if \( -a \in A \).
(343) \( A \subseteq B \) if and only if \( -A \subseteq -B \).
(344) \((\text{add inverse } G)^\circ A = -A\).
(345) \((\text{add inverse } G)^{-1}(A) = -A\).
(346) \(\text{add inverse } G \) is one-to-one. The theorem is a consequence of (9).
(347) \(\text{rng add inverse } G = \text{the carrier of } G \).

Let \( G \) be an additive group. One can verify that \(\text{add inverse } G \) is one-to-one and onto.

Now we state the propositions:

(348) \((\text{add inverse } G)^{-1} = \text{add inverse } G\).
(349) \((\text{The addition of } H)^\circ (P \times Q) = P + Q\).

Let \( G \) be a non empty additive magma and \( a \) be an element of \( G \). The functors: \( a^+ \) and \( ^+a \) yielding functions from \( G \) into \( G \) are defined by conditions,

(Def. 44) for every element \( x \) of \( G \), \( a^+(x) = a + x \),
(Def. 45) for every element \( x \) of \( G \), \( ^+a(x) = x + a \),

respectively. Let \( G \) be an additive group. One can verify that \( a^+ \) is one-to-one and onto and \( ^+a \) is one-to-one and onto.

Now we state the propositions:

(350) \((h^+)\circ P = h + P \). The theorem is a consequence of (73).
(351) \((^+h)\circ P = P + h \). The theorem is a consequence of (74).
(352) \((a^+)^{-1} = (-a)^+ \).
(353) \((^+a)^{-1} = ^+(-a) \).

We consider topological additive group structures which extend additive magmas and topological structures and are systems

\[ \{\text{a carrier, an addition, a topology}\} \]
where the carrier is a set, the addition is a binary operation on the carrier, the
topology is a family of subsets of the carrier.

Let $A$ be a non empty set, $R$ be a binary operation on $A$, and $T$ be a family
of subsets of $A$. Let us observe that $\langle A, R, T \rangle$ is non empty.

Let $x$ be a set, $R$ be a binary operation on $\{x\}$, and $T$ be a family of
subsets of $\{x\}$. Observe that $\langle \{x\}, R, T \rangle$ is trivial and every 1-element additive
magma is additive group-like, add-associative, and Abelian and there exists a
topological additive group structure which is strict and non empty and there
exists a topological additive group structure which is strict, topological space-
like, and 1-element.

Let $G$ be an additive group-like, add-associative, non empty topological
additive group structure. We say that $G$ is inverse-continuous if and only if

(Def. 46) \text{add inverse } G \text{ is continuous.}

Let $G$ be a topological space-like topological additive group structure. We
say that $G$ is continuous if and only if

(Def. 47) \text{for every function } f \text{ from } G \times G \text{ into } G \text{ such that } f = \text{the addition of}
G \text{ holds } f \text{ is continuous.}

One can check that there exists a topological space-like, additive group-like,
add-associative, 1-element topological additive group structure which is strict,
Abelian, inverse-continuous, and continuous.

A semi additive topological group is a topological space-like, additive group-
like, add-associative, non empty topological additive group structure.

A topological additive group is an inverse-continuous, continuous semi ad-
ditive topological group. Now we state the propositions:

(354) Let us consider a continuous, non empty, topological space-like topo-
logical additive group structure $T$, elements $a, b$ of $T$, and a neighbourhood
$W$ of $a + b$. Then there exists an open neighbourhood $A$ of $a$ and there
exists an open neighbourhood $B$ of $b$ such that $A + B \subseteq W$.

(355) Let us consider a topological space-like, non empty topological additive
group structure $T$. Suppose for every elements $a, b$ of $T$ for every neighbo-
ourhood $W$ of $a + b$, there exists a neighbourhood $A$ of $a$ and there exists
a neighbourhood $B$ of $b$ such that $A + B \subseteq W$. Then $T$ is continuous.

PROOF: For every point $W$ of $T \times T$ and for every neighbourhood $G$ of
$f(W)$, there exists a neighbourhood $H$ of $W$ such that $f^o H \subseteq G$ by [32]
(10)], (349). □

(356) Let us consider an inverse-continuous semi additive topological group $T$,
an element $a$ of $T$, and a neighbourhood $W$ of $-a$. Then there exists an
open neighbourhood $A$ of $a$ such that $-A \subseteq W$.

(357) Let us consider a semi additive topological group $T$. Suppose for every
element $a$ of $T$ for every neighbourhood $W$ of $-a$, there exists a neighbourhood $A$ of $a$ such that $-A \subseteq W$. Then $T$ is inverse-continuous. The theorem is a consequence of (344).

(358) Let us consider a topological additive group $T$, elements $a$, $b$ of $T$, and a neighbourhood $W$ of $a+(-b)$. Then there exists an open neighbourhood $A$ of $a$ and there exists an open neighbourhood $B$ of $b$ such that $A+(-B) \subseteq W$. Then $T$ is inverse-continuous. The theorem is a consequence of (354) and (356).

(359) Let us consider a semi additive topological group $T$. Suppose for every elements $a$, $b$ of $T$ for every neighbourhood $W$ of $a+(-b)$, there exists a neighbourhood $A$ of $a$ and there exists a neighbourhood $B$ of $b$ such that $A+(-B) \subseteq W$. Then $T$ is a topological additive group.

Proof: For every element $a$ of $T$ and for every neighbourhood $W$ of $-a$, there exists a neighbourhood $A$ of $a$ such that $-A \subseteq W$ by [28, (4)]. For every elements $a$, $b$ of $T$ and for every neighbourhood $W$ of $a+b$, there exists a neighbourhood $A$ of $a$ and there exists a neighbourhood $B$ of $b$ such that $A+B \subseteq W$. □

Let $G$ be a continuous, non empty, topological space-like topological additive group structure and $a$ be an element of $G$. One can check that $a^+$ is continuous and $^+a$ is continuous.

Let us consider a continuous semi additive topological group $G$ and an element $a$ of $G$. Now we state the propositions:

(360) $a^+$ is a homeomorphism of $G$. The theorem is a consequence of (352).

(361) $^+a$ is a homeomorphism of $G$. The theorem is a consequence of (353).

Let $G$ be a continuous semi additive topological group and $a$ be an element of $G$. The functors: $a^+$ and $^+a$ yield homeomorphisms of $G$. Now we state the proposition:

(362) Let us consider an inverse-continuous semi additive topological group $G$. Then add inverse $G$ is a homeomorphism of $G$. The theorem is a consequence of (348).

Let $G$ be an inverse-continuous semi additive topological group. Let us note that the functor add inverse $G$ yields a homeomorphism of $G$. Let us note that every semi additive topological group which is continuous is also homogeneous.

Let us consider a continuous semi additive topological group $G$, a closed subset $F$ of $G$, and an element $a$ of $G$. Now we state the propositions:

(363) $F+a$ is closed. The theorem is a consequence of (351).

(364) $a+F$ is closed. The theorem is a consequence of (350).

Let $G$ be a continuous semi additive topological group, $F$ be a closed subset of $G$, and $a$ be an element of $G$. Let us note that $F+a$ is closed and $a+F$ is
Let $G$ be an inverse-continuous semi additive topological group and $F$ be a closed subset of $G$. One can verify that $-F$ is closed.

Let us consider a continuous semi additive topological group $G$, an open subset $O$ of $G$, and an element $a$ of $G$. One can check that $A + a$ is open and $a + A$ is open.

Now we state the proposition:

(366) $O + a$ is open. The theorem is a consequence of (351).

(367) $a + O$ is open. The theorem is a consequence of (350).

Let $G$ be a continuous semi additive topological group, $A$ be an open subset of $G$, and $a$ be an element of $G$. One can check that $A + a$ is open and $a + A$ is open.

Now we state the proposition:

(368) $O + A$ is open. $O + A$ is open. The theorem is a consequence of (344).

Let $G$ be an inverse-continuous semi additive topological group and $A$ be an open subset of $G$. Observe that $-A$ is open.

Let us consider a continuous semi additive topological group $G$ and subsets $A, O$ of $G$.

Let us assume that $O$ is open. Now we state the propositions:

(369) $O + A$ is open.

Proof: $\text{Int}(O + A) = O + A$ by [48, (16)], (74), [48, (22)]. $\square$

(370) $A + O$ is open.

Proof: $\text{Int}(A + O) = A + O$ by [48, (16)], (73), [48, (22)]. $\square$

Let $G$ be a continuous semi additive topological group, $A$ be an open subset of $G$, and $B$ be a subset of $G$. Note that $A + B$ is open and $B + A$ is open.

Now we state the propositions:

(371) Let us consider an inverse-continuous semi additive topological group $G$, a point $a$ of $G$, and a neighbourhood $A$ of $a$. Then $-A$ is a neighbourhood of $-a$. The theorem is a consequence of (343).

(372) Let us consider a topological additive group $G$, a point $a$ of $G$, and a neighbourhood $A$ of $a + -a$. Then there exists an open neighbourhood $B$ of $a$ such that $B + -B \subseteq A$. The theorem is a consequence of (358) and (342).
Let us consider an inverse-continuous semi additive topological group \( G \), and a dense subset \( A \) of \( G \). Then \(-A\) is dense. The theorem is a consequence of (345).

Let \( G \) be an inverse-continuous semi additive topological group and \( A \) be a dense subset of \( G \). Observe that \(-A\) is dense.

Let us consider a continuous semi additive topological group \( G \), a dense subset \( A \) of \( G \), and a point \( a \) of \( G \). Now we state the propositions:

1. \( a + A \) is dense. The theorem is a consequence of (350).
2. \( A + a \) is dense. The theorem is a consequence of (351).

Let \( G \) be a continuous semi additive topological group, \( A \) be a dense subset of \( G \), and \( a \) be a point of \( G \). Let us observe that \( A + a \) is dense and \( a + A \) is dense.

Now we state the proposition:

1. \( \{ V + x, \text{where } V \text{ is a subset of } G, x \text{ is a point of } G : V \in B \text{ and } x \in M \} \) is a basis of \( G \).

**Proof:** Set \( Z = \{ V + x, \text{where } V \text{ is a subset of } G, x \text{ is a point of } G : V \in B \text{ and } x \in M \} \). \( Z \subseteq \) the topology of \( G \) by \( [38, (12)] \). For every subset \( W \) of \( G \) such that \( W \) is open for every point \( a \) of \( G \) such that \( a \in W \) there exists a subset \( V \) of \( G \) such that \( V \subseteq Z \) and \( a \in V \) and \( V \subseteq W \) by (8), \( [28, (3)], (74), (372) \). \( Z \subseteq 2^\alpha \), where \( \alpha \) is the carrier of \( G \). \( \square \)

One can check that every topological additive group is regular.

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**References**


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