

# Product Pre-Measure

Noboru Endou  
Gifu National College of Technology  
Gifu, Japan

**Summary.** In this article we formalize in Mizar [5] product pre-measure on product sets of measurable sets. Although there are some approaches to construct product measure [22], [6], [9], [21], [25], we start it from  $\sigma$ -measure because existence of  $\sigma$ -measure on any semialgebras has been proved in [15]. In this approach, we use some theorems for integrals.

MSC: 28A35 03B35

Keywords: product measure; pre-measure

MML identifier: MEASUR10, version: 8.1.04 5.36.1267

## 1. PRELIMINARIES

Now we state the proposition:

- (1) Let us consider non empty sets  $A, A_1, A_2, B, B_1, B_2$ . Then  $A_1 \times B_1$  misses  $A_2 \times B_2$  and  $A \times B = A_1 \times B_1 \cup A_2 \times B_2$  if and only if  $A_1$  misses  $A_2$  and  $A = A_1 \cup A_2$  and  $B = B_1$  and  $B = B_2$  or  $B_1$  misses  $B_2$  and  $B = B_1 \cup B_2$  and  $A = A_1$  and  $A = A_2$ .

Let  $C, D$  be non empty sets,  $F$  be a sequence of  $D^C$ , and  $n$  be a natural number. One can check that the functor  $F(n)$  yields a function from  $C$  into  $D$ .

- (2) Let us consider sets  $X, Y, A, B$ , and objects  $x, y$ . Suppose  $x \in X$  and  $y \in Y$ . Then  $\chi_{A,X}(x) \cdot \chi_{B,Y}(y) = \chi_{A \times B, X \times Y}(x, y)$ .

Let  $A, B$  be sets. One can verify that  $\chi_{A,B}$  is non-negative.

- (3) Let us consider a non empty set  $X$ , a semialgebra  $S$  of sets of  $X$ , a pre-measure  $P$  of  $S$ , an induced measure  $m$  of  $S$  and  $P$ , and an induced  $\sigma$ -measure  $M$  of  $S$  and  $m$ . Then  $\text{COM}(M)$  is complete on  $\text{COM}(\sigma(\text{the field generated by } S), M)$ .

The functor  $\text{Intervals}_{\mathbb{R}}$  yielding a semialgebra of sets of  $\mathbb{R}$  is defined by the term

(Def. 1) the set of all  $I$  where  $I$  is an interval.

Now we state the propositions:

(4)  $\text{Halflines} \subseteq \text{Intervals}_{\mathbb{R}}$ .

(5) Let us consider a subset  $I$  of  $\mathbb{R}$ . If  $I$  is an interval, then  $I \in$  the Borel sets.

(6) (i)  $\sigma(\text{Intervals}_{\mathbb{R}}) =$  the Borel sets, and

(ii)  $\sigma(\text{the field generated by } \text{Intervals}_{\mathbb{R}}) =$  the Borel sets.

The theorem is a consequence of (4) and (5).

## 2. FAMILY OF SEMIALGEBRAS, FIELDS AND MEASURES

Now we state the propositions:

(7) Let us consider sets  $X_1, X_2$ , a non empty family  $S_1$  of subsets of  $X_1$ , and a non empty family  $S_2$  of subsets of  $X_2$ . Then the set of all  $a \times b$  where  $a$  is an element of  $S_1$ ,  $b$  is an element of  $S_2$  is a non empty family of subsets of  $X_1 \times X_2$ .

(8) Let us consider sets  $X, Y$ , a family  $M$  of subsets of  $X$  with the empty element, and a family  $N$  of subsets of  $Y$  with the empty element. Then the set of all  $A \times B$  where  $A$  is an element of  $M$ ,  $B$  is an element of  $N$  is a family of subsets of  $X \times Y$  with the empty element. The theorem is a consequence of (7).

(9) Let us consider a set  $X$ , and disjoint valued finite sequences  $O, T$  of elements of  $X$ . Suppose  $\bigcup \text{rng } O$  misses  $\bigcup \text{rng } T$ . Then  $O \cap T$  is a disjoint valued finite sequence of elements of  $X$ .

(10) Let us consider sets  $X_1, X_2$ , a semiring  $S_1$  of  $X_1$ , and a semiring  $S_2$  of  $X_2$ . Then the set of all  $A \times B$  where  $A$  is an element of  $S_1$ ,  $B$  is an element of  $S_2$  is a semiring of  $X_1 \times X_2$ .

(11) Let us consider sets  $X_1, X_2$ , a semialgebra  $S_1$  of sets of  $X_1$ , and a semialgebra  $S_2$  of sets of  $X_2$ . Then the set of all  $A \times B$  where  $A$  is an element of  $S_1$ ,  $B$  is an element of  $S_2$  is a semialgebra of sets of  $X_1 \times X_2$ . The theorem is a consequence of (10).

(12) Let us consider sets  $X_1, X_2$ , a field  $O$  of subsets of  $X_1$ , and a field  $T$  of subsets of  $X_2$ . Then the set of all  $A \times B$  where  $A$  is an element of  $O$ ,  $B$  is an element of  $T$  is a semialgebra of sets of  $X_1 \times X_2$ . The theorem is a consequence of (11).

Let  $n$  be a non zero natural number and  $X$  be a non-empty,  $n$ -element finite sequence.

A family of semialgebras of  $X$  is an  $n$ -element finite sequence and is defined by

(Def. 2) for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $it(i)$  is a semialgebra of sets of  $X(i)$ .

Let us observe that a family of semialgebras of  $X$  is a  $\cap$ -closed yielding family of semirings of  $X$ . Now we state the proposition:

(13) Let us consider a non zero natural number  $n$ , a non-empty,  $n$ -element finite sequence  $X$ , a family  $S$  of semialgebras of  $X$ , and a natural number  $i$ . If  $i \in \text{Seg } n$ , then  $X(i) \in S(i)$ .

Let us consider a non-empty, 1-element finite sequence  $X$  and a family  $S$  of semialgebras of  $X$ . Now we state the propositions:

(14) the set of all  $\prod \langle s \rangle$  where  $s$  is an element of  $S(1)$  is a semialgebra of sets of the set of all  $\langle x \rangle$  where  $x$  is an element of  $X(1)$ . The theorem is a consequence of (13).

(15)  $\text{SemiringProduct}(S)$  is a semialgebra of sets of  $\prod X$ . The theorem is a consequence of (14).

(16) Let us consider sets  $X_1, X_2$ , a semialgebra  $S_1$  of sets of  $X_1$ , and a semialgebra  $S_2$  of sets of  $X_2$ . Then the set of all  $s_1 \times s_2$  where  $s_1$  is an element of  $S_1$ ,  $s_2$  is an element of  $S_2$  is a semialgebra of sets of  $X_1 \times X_2$ .

(17) Let us consider a non zero natural number  $n$ , a non-empty,  $n$ -element finite sequence  $X$ , and a family  $S$  of semialgebras of  $X$ . Then  $\text{SemiringProduct}(S)$  is a semialgebra of sets of  $\prod X$ .

PROOF: Define  $\mathcal{P}[\text{non zero natural number}] \equiv$  for every non-empty,  $\mathbb{N}$ -element finite sequence  $X$  for every family  $S$  of semialgebras of  $X$ ,  $\text{SemiringProduct}(S)$  is a semialgebra of sets of  $\prod X$ .  $\mathcal{P}[1]$ . For every non zero natural number  $k$ ,  $\mathcal{P}[k]$  from [3, Sch. 10].  $\square$

(18) Let us consider a non zero natural number  $n$ , a non-empty,  $n$ -element finite sequence  $X_8$ , a non-empty, 1-element finite sequence  $X_1$ , a family  $S_4$  of semialgebras of  $X_8$ , and a family  $S_1$  of semialgebras of  $X_1$ . Then  $\text{SemiringProduct}(S_4 \cap S_1)$  is a semialgebra of sets of  $\prod(X_8 \cap X_1)$ . The theorem is a consequence of (17), (16), and (13).

Let  $n$  be a non zero natural number and  $X$  be a non-empty,  $n$ -element finite sequence.

A family of fields of  $X$  is an  $n$ -element finite sequence and is defined by

(Def. 3) for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $it(i)$  is a field of subsets of  $X(i)$ .

Let  $S$  be a family of fields of  $X$  and  $i$  be a natural number. Assume  $i \in \text{Seg } n$ . Observe that the functor  $S(i)$  yields a field of subsets of  $X(i)$ .

Observe that a family of fields of  $X$  is a family of semialgebras of  $X$ .

Let us consider a non-empty, 1-element finite sequence  $X$  and a family  $S$  of fields of  $X$ . Now we state the propositions:

- (19) the set of all  $\prod \langle s \rangle$  where  $s$  is an element of  $S(1)$  is a field of subsets of the set of all  $\langle x \rangle$  where  $x$  is an element of  $X(1)$ . The theorem is a consequence of (14).
- (20)  $\text{SemiringProduct}(S)$  is a field of subsets of  $\prod X$ . The theorem is a consequence of (19).

Let  $n$  be a non zero natural number,  $X$  be a non-empty,  $n$ -element finite sequence, and  $S$  be a family of fields of  $X$ .

A family of measures of  $S$  is an  $n$ -element finite sequence and is defined by

- (Def. 4) for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $it(i)$  is a measure on  $S(i)$ .

### 3. PRODUCT OF TWO MEASURES

Let  $X_1, X_2$  be sets,  $S_1$  be a field of subsets of  $X_1$ , and  $S_2$  be a field of subsets of  $X_2$ . The functor  $\text{MeasRect}(S_1, S_2)$  yielding a semialgebra of sets of  $X_1 \times X_2$  is defined by the term

- (Def. 5) the set of all  $A \times B$  where  $A$  is an element of  $S_1$ ,  $B$  is an element of  $S_2$ .

Now we state the proposition:

- (21) Let us consider a set  $X$ , and a field  $F$  of subsets of  $X$ . Then there exists a semialgebra  $S$  of sets of  $X$  such that
- (i)  $F = S$ , and
- (ii)  $F$  = the field generated by  $S$ .

Let  $X_1, X_2$  be sets,  $S_1$  be a field of subsets of  $X_1$ ,  $S_2$  be a field of subsets of  $X_2$ ,  $m_1$  be a measure on  $S_1$ , and  $m_2$  be a measure on  $S_2$ . The functor  $\text{ProdpreMeas}(m_1, m_2)$  yielding a non-negative, zeroed function from  $\text{MeasRect}(S_1, S_2)$  into  $\overline{\mathbb{R}}$  is defined by

- (Def. 6) for every element  $C$  of  $\text{MeasRect}(S_1, S_2)$ , there exists an element  $A$  of  $S_1$  and there exists an element  $B$  of  $S_2$  such that  $C = A \times B$  and  $it(C) = m_1(A) \cdot m_2(B)$ .

Now we state the propositions:

- (22) Let us consider sets  $X_1, X_2$ , a field  $S_1$  of subsets of  $X_1$ , a field  $S_2$  of subsets of  $X_2$ , a measure  $m_1$  on  $S_1$ , a measure  $m_2$  on  $S_2$ , and sets  $A, B$ .

Suppose  $A \in S_1$  and  $B \in S_2$ . Then  $(\text{ProdpreMeas}(m_1, m_2))(A \times B) = m_1(A) \cdot m_2(B)$ .

- (23) Let us consider sets  $X_1, X_2$ , a non empty family  $S_1$  of subsets of  $X_1$ , a non empty family  $S_2$  of subsets of  $X_2$ , a non empty family  $S$  of subsets of  $X_1 \times X_2$ , and a finite sequence  $H$  of elements of  $S$ . Suppose  $S =$  the set of all  $A \times B$  where  $A$  is an element of  $S_1, B$  is an element of  $S_2$ . Then there exists a finite sequence  $F$  of elements of  $S_1$  and there exists a finite sequence  $G$  of elements of  $S_2$  such that  $\text{len } H = \text{len } F$  and  $\text{len } H = \text{len } G$  and for every natural number  $k$  such that  $k \in \text{dom } H$  and  $H(k) \neq \emptyset$  holds  $H(k) = F(k) \times G(k)$ .

PROOF: For every natural number  $k$  such that  $k \in \text{dom } H$  there exists an element  $A$  of  $S_1$  and there exists an element  $B$  of  $S_2$  such that  $H(k) = A \times B$ . Define  $\mathcal{P}[\text{natural number, set}] \equiv$  there exists an element  $B$  of  $S_2$  such that  $H(\$1) = \$2 \times B$ . Consider  $F$  being a finite sequence of elements of  $S_1$  such that  $\text{dom } F = \text{Seg len } H$  and for every natural number  $k$  such that  $k \in \text{Seg len } H$  holds  $\mathcal{P}[k, F(k)]$  from [4, Sch. 5]. Define  $\mathcal{Q}[\text{natural number, set}] \equiv$  there exists an element  $A$  of  $S_1$  such that  $H(\$1) = A \times \$2$ . For every natural number  $k$  such that  $k \in \text{Seg len } H$  there exists an element  $B$  of  $S_2$  such that  $\mathcal{Q}[k, B]$ . Consider  $G$  being a finite sequence of elements of  $S_2$  such that  $\text{dom } G = \text{Seg len } H$  and for every natural number  $k$  such that  $k \in \text{Seg len } H$  holds  $\mathcal{Q}[k, G(k)]$  from [4, Sch. 5].  $\square$

- (24) Let us consider a set  $X$ , a non empty, semi-diff-closed,  $\cap$ -closed family  $S$  of subsets of  $X$ , and elements  $E_1, E_2$  of  $S$ . Then there exist disjoint valued finite sequences  $O, T, F$  of elements of  $S$  such that
- (i)  $\bigcup \text{rng } O = E_1 \setminus E_2$ , and
  - (ii)  $\bigcup \text{rng } T = E_2 \setminus E_1$ , and
  - (iii)  $\bigcup \text{rng } F = E_1 \cap E_2$ , and
  - (iv)  $(O \cap T) \cap F$  is a disjoint valued finite sequence of elements of  $S$ .

The theorem is a consequence of (9).

- (25) Let us consider sets  $X_1, X_2$ , a field  $S_1$  of subsets of  $X_1$ , a field  $S_2$  of subsets of  $X_2$ , a measure  $m_1$  on  $S_1$ , a measure  $m_2$  on  $S_2$ , and elements  $E_1, E_2$  of  $\text{MeasRect}(S_1, S_2)$ . Suppose  $E_1$  misses  $E_2$  and  $E_1 \cup E_2 \in \text{MeasRect}(S_1, S_2)$ . Then  $(\text{ProdpreMeas}(m_1, m_2))(E_1 \cup E_2) = (\text{ProdpreMeas}(m_1, m_2))(E_1) + (\text{ProdpreMeas}(m_1, m_2))(E_2)$ . The theorem is a consequence of (1) and (22).
- (26) Let us consider a non empty set  $X$ , a non empty family  $S$  of subsets of  $X$ , a function  $f$  from  $\mathbb{N}$  into  $S$ , and a sequence  $F$  of partial functions from  $X$  into  $\overline{\mathbb{R}}$ . Suppose  $f$  is disjoint valued and for every natural number

$n$ ,  $F(n) = \chi_{f(n), X}$ . Let us consider an object  $x$ . Suppose  $x \in X$ . Then  $\chi_{\bigcup_{f, X}}(x) = (\lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})(x)$ .

- (27) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ , and a real number  $r$ . Suppose  $\text{dom } f \in S$  and  $0 \leq r$  and for every object  $x$  such that  $x \in \text{dom } f$  holds  $f(x) = r$ . Then  $\int f \, dM = r \cdot M(\text{dom } f)$ .

Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ , and an element  $A$  of  $S$ . Now we state the propositions:

- (28) Suppose there exists an element  $E$  of  $S$  such that  $E = \text{dom } f$  and  $f$  is measurable on  $E$  and for every object  $x$  such that  $x \in \text{dom } f \setminus A$  holds  $f(x) = 0$  and  $f$  is non-negative. Then  $\int f \, dM = \int f \upharpoonright A \, dM$ . The theorem is a consequence of (27).
- (29) If  $f$  is integrable on  $M$  and for every object  $x$  such that  $x \in \text{dom } f \setminus A$  holds  $f(x) = 0$ , then  $\int f \, dM = \int f \upharpoonright A \, dM$ . The theorem is a consequence of (27).
- (30) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_2$  on  $S_2$ , a function  $D$  from  $\mathbb{N}$  into  $S_1$ , a function  $E$  from  $\mathbb{N}$  into  $S_2$ , an element  $A$  of  $S_1$ , an element  $B$  of  $S_2$ , a sequence  $F$  of partial functions from  $X_2$  into  $\overline{\mathbb{R}}$ , a sequence  $R$  of  $\mathbb{R}^{X_1}$ , and an element  $x$  of  $X_1$ . Suppose for every natural number  $n$ ,  $R(n) = \chi_{D(n), X_1}$  and for every natural number  $n$ ,  $F(n) = R(n)(x) \cdot \chi_{E(n), X_2}$  and for every natural number  $n$ ,  $E(n) \subseteq B$ . Then there exists a sequence  $I$  of extended reals such that

- (i) for every natural number  $n$ ,  $I(n) = M_2(E(n)) \cdot \chi_{D(n), X_1}(x)$ , and
- (ii)  $I$  is summable, and
- (iii)  $\int \lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}} \, dM_2 = \sum I$ .

PROOF: For every natural number  $n$ ,  $\text{dom}(F(n)) = X_2$ . Reconsider  $S_3 = X_2$  as an element of  $S_2$ . For every natural number  $n$  and for every set  $y$  such that  $y \in E(n)$  holds  $F(n)(y) = 0$  or  $F(n)(y) = 1$  by [10, (3)], [18, (1)], [12, (39)]. For every natural number  $n$  and for every set  $y$  such that  $y \notin E(n)$  holds  $F(n)(y) = 0$ . For every natural number  $n$ ,  $F(n)$  is non-negative and  $F(n)$  is measurable on  $B$  by [8, (51)], [17, (37)], [18, (29)]. For every element  $y$  of  $X_2$  such that  $y \in B$  holds  $F \# y$  is summable by [8, (51), (39)], [19, (16)], [29, (37)].

Consider  $I$  being a sequence of extended reals such that for every natural number  $n$ ,  $I(n) = \int F(n) \upharpoonright B \, dM_2$  and  $I$  is summable and  $\int \lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}} \upharpoonright B \, dM_2 = \sum I$ . For every natural number  $n$ ,  $I(n) =$

$M_2(E(n)) \cdot \chi_{D(n), X_1}(x)$  by [28, (61)], [10, (47), (49)], [18, (29)]. For every natural number  $n$ ,  $F(n)$  is measurable on  $S_3$  by [18, (29)], [17, (37)]. For every natural number  $n$ ,  $F(n)$  is without  $-\infty$ . For every element  $y$  of  $X_2$  such that  $y \in S_3$  holds  $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}} \# y$  is convergent by [19, (38)]. For every object  $y$  such that  $y \in \text{dom} \lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}} \setminus B$  holds  $(\lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})(y) = 0$  by [19, (43)], [16, (52)]. For every object  $y$  such that  $y \in \text{dom} \lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}$  holds  $(\lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})(y) \geq 0$  by [19, (36)], [8, (51)], [19, (10), (38)].  $\square$

- (31) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , an element  $A$  of  $S$ , and an extended real number  $p$ . Then  $X \mapsto p$  is measurable on  $A$ .  
 PROOF: For every real number  $r$ ,  $A \cap \text{GTE-dom}(X \mapsto p, r) \in S$  by [26, (7)], [7, (7)].  $\square$

Let  $A, X$  be sets. The functor  $\bar{\chi}_{A, X}$  yielding a function from  $X$  into  $\bar{\mathbb{R}}$  is defined by

- (Def. 7) for every object  $x$  such that  $x \in X$  holds if  $x \in A$ , then  $it(x) = +\infty$  and if  $x \notin A$ , then  $it(x) = 0$ .

Now we state the proposition:

- (32) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , and elements  $A, B$  of  $S$ . Then  $\bar{\chi}_{A, X}$  is measurable on  $B$ .

Let  $X, A$  be sets. Let us observe that  $\bar{\chi}_{A, X}$  is non-negative.

- (33) Let us consider a non empty set  $X$ , a  $\sigma$ -field  $S$  of subsets of  $X$ , a  $\sigma$ -measure  $M$  on  $S$ , and an element  $A$  of  $S$ . Then

- (i) if  $M(A) \neq 0$ , then  $\int \bar{\chi}_{A, X} dM = +\infty$ , and
- (ii) if  $M(A) = 0$ , then  $\int \bar{\chi}_{A, X} dM = 0$ .

PROOF: Reconsider  $X_3 = X$  as an element of  $S$ . Reconsider  $X_2 = X_3 \setminus A$  as an element of  $S$ . Reconsider  $F = \bar{\chi}_{A, X} \upharpoonright A$  as a partial function from  $X$  to  $\bar{\mathbb{R}}$ . Reconsider  $O = \bar{\chi}_{A, X} \upharpoonright X_2$  as a partial function from  $X$  to  $\bar{\mathbb{R}}$ . Reconsider  $T = \bar{\chi}_{A, X} \upharpoonright (X_2 \cup A)$  as a partial function from  $X$  to  $\bar{\mathbb{R}}$ .  $\int F dM = 0$ .  $O$  is measurable on  $X_2$ . For every element  $x$  of  $X$  such that  $x \in \text{dom}(\bar{\chi}_{A, X} \upharpoonright X_2)$  holds  $(\bar{\chi}_{A, X} \upharpoonright X_2)(x) = 0$  by [10, (47)].  $\int T dM = \int O dM + 0$ .  $\square$

- (34) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_1$  on  $S_1$ , a  $\sigma$ -measure  $M_2$  on  $S_2$ , and a disjoint valued function  $K$  from  $\mathbb{N}$  into  $\text{MeasRect}(S_1, S_2)$ . Suppose  $\bigcup K \in \text{MeasRect}(S_1, S_2)$ . Then  $(\text{ProdpreMeas}(M_1, M_2))(\bigcup K) = \overline{\sum}(\text{ProdpreMeas}(M_1, M_2) \cdot K)$ .

PROOF: Consider  $A$  being an element of  $S_1$ ,  $B$  being an element of  $S_2$  such that  $\bigcup K = A \times B$ . Consider  $P$  being an element of  $S_1$ ,  $Q$  being an element of  $S_2$  such that  $\bigcup K = P \times Q$  and  $(\text{ProdpreMeas}(M_1, M_2))(\bigcup K) = M_1(P) \cdot$

$M_2(Q)$ . Define  $\mathcal{F}(\text{object}) = \chi_{K(\$_1), X_1 \times X_2}$ . Consider  $X_6$  being a sequence of partial functions from  $X_1 \times X_2$  into  $\overline{\mathbb{R}}$  such that for every natural number  $n$ ,  $X_6(n) = \mathcal{F}(n)$  from [24, Sch. 1]. Define  $\mathcal{P}[\text{natural number, object}] \equiv \$_2 = \pi_1(K(\$_1))$ . For every element  $i$  of  $\mathbb{N}$ , there exists an element  $A$  of  $S_1$  such that  $\mathcal{P}[i, A]$  by [2, (9)], [7, (7)]. Consider  $D$  being a function from  $\mathbb{N}$  into  $S_1$  such that for every element  $i$  of  $\mathbb{N}$ ,  $\mathcal{P}[i, D(i)]$  from [11, Sch. 3]. Define  $\mathcal{Q}[\text{natural number, object}] \equiv \$_2 = \pi_2(K(\$_1))$ . For every element  $i$  of  $\mathbb{N}$ , there exists an element  $B$  of  $S_2$  such that  $\mathcal{Q}[i, B]$  by [2, (9)], [7, (7)].

Consider  $E$  being a function from  $\mathbb{N}$  into  $S_2$  such that for every element  $i$  of  $\mathbb{N}$ ,  $\mathcal{Q}[i, E(i)]$  from [11, Sch. 3]. Define  $\mathcal{O}(\text{object}) = \chi_{D(\$_1), X_1}$ . Consider  $X_7$  being a sequence of partial functions from  $X_1$  into  $\overline{\mathbb{R}}$  such that for every natural number  $n$ ,  $X_7(n) = \mathcal{O}(n)$  from [24, Sch. 1]. Define  $\mathcal{T}(\text{object}) = \chi_{E(\$_1), X_2}$ . Consider  $X_4$  being a sequence of partial functions from  $X_2$  into  $\overline{\mathbb{R}}$  such that for every natural number  $n$ ,  $X_4(n) = \mathcal{T}(n)$  from [24, Sch. 1]. For every natural number  $n$  and for every objects  $x, y$  such that  $x \in X_1$  and  $y \in X_2$  holds  $X_6(n)(x, y) = X_7(n)(x) \cdot X_4(n)(y)$  by [14, (87)], [2, (9)], (2).  $(\text{ProdpreMeas}(M_1, M_2))(\cup K) = M_1(A) \cdot M_2(B)$  by [14, (110)]. Reconsider  $C_1 = \chi_{A \times B, X_1 \times X_2}$  as a function from  $X_1 \times X_2$  into  $\overline{\mathbb{R}}$ . For every element  $x$  of  $X_1$ ,  $M_2(B) \cdot \chi_{A, X_1}(x) = \int \text{curry}(C_1, x) dM_2$  by (2), [13, (5)], [19, (14)], [23, (4)]. For every object  $n$  such that  $n \in \mathbb{N}$  holds  $X_7(n) \in \mathbb{R}^{X_1}$  by [12, (39)]. Reconsider  $R_1 = X_7$  as a sequence of  $\mathbb{R}^{X_1}$ . For every natural number  $n$ ,  $D(n) \subseteq A$  and  $E(n) \subseteq B$  by [2, (10)], [1, (1)]. For every element  $x$  of  $X_1$ , there exists a sequence  $X_5$  of partial functions from  $X_2$  into  $\overline{\mathbb{R}}$  and there exists a sequence  $I$  of extended reals such that for every natural number  $n$ ,  $X_5(n) = R_1(n)(x) \cdot \chi_{E(n), X_2}$  and for every natural number  $n$ ,  $I(n) = M_2(E(n)) \cdot \chi_{D(n), X_1}(x)$  and  $I$  is summable and  $\int \lim(\sum_{\alpha=0}^{\kappa} X_5(\alpha))_{\kappa \in \mathbb{N}} dM_2 = \sum I$  by [13, (45)], (30).

Reconsider  $L_1 = \lim(\sum_{\alpha=0}^{\kappa} X_6(\alpha))_{\kappa \in \mathbb{N}}$  as a function from  $X_1 \times X_2$  into  $\overline{\mathbb{R}}$ . For every element  $x$  of  $X_1$  and for every element  $y$  of  $X_2$ ,  $(\text{curry}(C_1, x))(y) = (\text{curry}(L_1, x))(y)$ . For every element  $x$  of  $X_1$ ,  $\text{curry}(C_1, x) = \text{curry}(L_1, x)$ . For every element  $x$  of  $X_1$ ,  $M_2(B) \cdot \chi_{A, X_1}(x) = \int \text{curry}(L_1, x) dM_2$ . For every element  $x$  of  $X_1$ , there exists a sequence  $I$  of extended reals such that for every natural number  $n$ ,  $I(n) = M_2(E(n)) \cdot \chi_{D(n), X_1}(x)$  and  $M_2(B) \cdot \chi_{A, X_1}(x) = \sum I$  by [8, (51)], [19, (38), (29), (30)]. Define  $\mathcal{R}[\text{natural number, object}] \equiv \text{if } M_2(E(\$_1)) = +\infty$ , then  $\$_2 = \bar{\chi}_{D(\$_1), X_1}$  and if  $M_2(E(\$_1)) \neq +\infty$ , then there exists a real number  $m_2$  such that  $m_2 = M_2(E(\$_1))$  and  $\$_2 = m_2 \cdot \chi_{D(\$_1), X_1}$ . For every element  $n$  of  $\mathbb{N}$ , there exists an element  $y$  of  $X_1 \dot{\rightarrow} \overline{\mathbb{R}}$  such that  $\mathcal{R}[n, y]$  by [13, (45)], [8, (51)]. Consider  $F_1$  being a function from  $\mathbb{N}$  into  $X_1 \dot{\rightarrow} \overline{\mathbb{R}}$  such that for every element  $n$  of  $\mathbb{N}$ ,  $\mathcal{R}[n, F_1(n)]$  from [11, Sch. 3]. For every natural number

$n$ ,  $\text{dom}(F_1(n)) = X_1$ . For every natural number  $n$ ,  $F_1(n)$  is non-negative by [8, (51)]. For every natural numbers  $n, m$ ,  $\text{dom}(F_1(n)) = \text{dom}(F_1(m))$ .

Reconsider  $X_3 = X_1$  as an element of  $S_1$ . For every natural number  $n$ ,  $F_1(n)$  is non-negative and  $F_1(n)$  is measurable on  $A$  and  $F_1(n)$  is measurable on  $X_3$  by (32), [18, (29)], [17, (37)]. For every element  $x$  of  $X_1$  such that  $x \in A$  holds  $F_1 \# x$  is summable by [8, (51), (39)], [20, (2)]. Consider  $J$  being a sequence of extended reals such that for every natural number  $n$ ,  $J(n) = \int F_1(n) \upharpoonright A \, dM_1$  and  $J$  is summable and  $\int \lim(\sum_{\alpha=0}^{\kappa} F_1(\alpha))_{\kappa \in \mathbb{N}} \upharpoonright A \, dM_1 = \sum J$ . For every natural number  $n$ ,  $J(n) = \int F_1(n) \, dM_1$ . Reconsider  $X_3 = X_1$  as an element of  $S_1$ . For every element  $n$  of  $\mathbb{N}$ ,  $J(n) = (\text{ProdpreMeas}(M_1, M_2) \cdot K)(n)$  by (33), [8, (51)], [18, (29)], [16, (86), (88)]. For every element  $x$  of  $X_1$ ,  $(\lim(\sum_{\alpha=0}^{\kappa} F_1(\alpha))_{\kappa \in \mathbb{N}})(x) \geq 0$  by [19, (38)], [29, (37), (23)], [8, (51)]. For every natural number  $n$ ,  $F_1(n)$  is measurable on  $X_3$  and  $F_1(n)$  is without  $-\infty$ . For every object  $x$  such that  $x \in \text{dom} \lim(\sum_{\alpha=0}^{\kappa} F_1(\alpha))_{\kappa \in \mathbb{N}} \setminus A$  holds  $(\lim(\sum_{\alpha=0}^{\kappa} F_1(\alpha))_{\kappa \in \mathbb{N}})(x) = 0$  by [19, (30), (32)], [16, (52)].  $\int \lim(\sum_{\alpha=0}^{\kappa} F_1(\alpha))_{\kappa \in \mathbb{N}} \, dM_1 = \int \lim(\sum_{\alpha=0}^{\kappa} F_1(\alpha))_{\kappa \in \mathbb{N}} \upharpoonright A \, dM_1$ .  $\int \lim(\sum_{\alpha=0}^{\kappa} F_1(\alpha))_{\kappa \in \mathbb{N}} \, dM_1 = M_1(A) \cdot M_2(B)$  by [11, (63)], [19, (30), (32)], [8, (51)].  $\square$

- (35) Let us consider a without  $-\infty$  finite sequence  $f$  of elements of  $\overline{\mathbb{R}}$ , and a without  $-\infty$  sequence  $s$  of extended reals. Suppose for every object  $n$  such that  $n \in \text{dom} f$  holds  $f(n) = s(n)$ .

Then  $\sum f + s(0) = (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(\text{len} f)$ .

PROOF: Consider  $F$  being a sequence of  $\overline{\mathbb{R}}$  such that  $\sum f = F(\text{len} f)$  and  $F(0) = 0$  and for every natural number  $i$  such that  $i < \text{len} f$  holds  $F(i+1) = F(i) + f(i+1)$ . Define  $\mathcal{P}[\text{natural number}] \equiv$  if  $\$1 \leq \text{len} f$ , then  $F(\$1) + s(0) = (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(\$1)$  and  $F(\$1) \neq -\infty$ . For every natural number  $k$  such that  $\mathcal{P}[k]$  holds  $\mathcal{P}[k+1]$  by [3, (11)], [27, (25)], [16, (10)], [3, (13)]. For every natural number  $k$ ,  $\mathcal{P}[k]$  from [3, Sch. 2].  $\square$

- (36) Let us consider a non-negative finite sequence  $f$  of elements of  $\overline{\mathbb{R}}$ , and a sequence  $s$  of extended reals. Suppose for every object  $n$  such that  $n \in \text{dom} f$  holds  $f(n) = s(n)$  and for every element  $n$  of  $\mathbb{N}$  such that  $n \notin \text{dom} f$  holds  $s(n) = 0$ . Then

(i)  $\sum f = \sum s$ , and

(ii)  $\sum f = \overline{\sum} s$ .

PROOF: For every object  $n$  such that  $n \in \text{dom} s$  holds  $0 \leq s(n)$  by [8, (51)].  $\sum f + s(0) = (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(\text{len} f)$ . Define  $\mathcal{P}[\text{natural number}] \equiv (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(\text{len} f) = ((\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}} \upharpoonright \text{len} f)(\$1)$ . For every natural number  $k$  such that  $\mathcal{P}[k]$  holds  $\mathcal{P}[k+1]$  by [27, (25)]. For every natural number  $k$ ,  $\mathcal{P}[k]$  from [3, Sch. 2].  $\square$

- (37) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_1$  on  $S_1$ , a  $\sigma$ -measure  $M_2$  on  $S_2$ , and a disjoint valued finite sequence  $F$  of elements of  $\text{MeasRect}(S_1, S_2)$ . Suppose  $\bigcup F \in \text{MeasRect}(S_1, S_2)$ . Then  $(\text{ProdpreMeas}(M_1, M_2))(\bigcup F) = \sum(\text{ProdpreMeas}(M_1, M_2) \cdot F)$ .

PROOF: Set  $S = \text{MeasRect}(S_1, S_2)$ . Define  $\mathcal{P}[\text{object}, \text{object}] \equiv \text{if } \$1 \in \text{dom } F, \text{ then } \$2 = F(\$1) \text{ and if } \$1 \notin \text{dom } F, \text{ then } \$2 = \emptyset$ . For every element  $n$  of  $\mathbb{N}$ , there exists an element  $y$  of  $S$  such that  $\mathcal{P}[n, y]$  by [10, (3)]. Consider  $G$  being a function from  $\mathbb{N}$  into  $S$  such that for every element  $n$  of  $\mathbb{N}$ ,  $\mathcal{P}[n, G(n)]$  from [11, Sch. 3]. For every object  $x$  such that  $x \notin \text{dom } F$  holds  $G(x) = \emptyset$ . For every objects  $x, y$  such that  $x \neq y$  holds  $G(x)$  misses  $G(y)$ .  $(\text{ProdpreMeas}(M_1, M_2))(\bigcup F) = \overline{\sum}(\text{ProdpreMeas}(M_1, M_2) \cdot G)$ . For every object  $n$  such that  $n \in \text{dom}(\text{ProdpreMeas}(M_1, M_2) \cdot F)$  holds  $(\text{ProdpreMeas}(M_1, M_2) \cdot F)(n) = (\text{ProdpreMeas}(M_1, M_2) \cdot G)(n)$  by [10, (11), (12), (13)]. For every element  $n$  of  $\mathbb{N}$  such that  $n \notin \text{dom}(\text{ProdpreMeas}(M_1, M_2) \cdot F)$  holds  $(\text{ProdpreMeas}(M_1, M_2) \cdot G)(n) = 0$  by [10, (3), (11), (13)].  $\square$

- (38) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_1$  on  $S_1$ , and a  $\sigma$ -measure  $M_2$  on  $S_2$ . Then  $\text{ProdpreMeas}(M_1, M_2)$  is a pre-measure of  $\text{MeasRect}(S_1, S_2)$ . The theorem is a consequence of (37) and (34).

Let  $X_1, X_2$  be non empty sets,  $S_1$  be a  $\sigma$ -field of subsets of  $X_1$ ,  $S_2$  be a  $\sigma$ -field of subsets of  $X_2$ ,  $M_1$  be a  $\sigma$ -measure on  $S_1$ , and  $M_2$  be a  $\sigma$ -measure on  $S_2$ . Let us observe that the functor  $\text{ProdpreMeas}(M_1, M_2)$  yields a pre-measure of  $\text{MeasRect}(S_1, S_2)$ .

## REFERENCES

- [1] Grzegorz Bancerek. Towards the construction of a model of Mizar concepts. *Formalized Mathematics*, 16(2):207–230, 2008. doi:10.2478/v10037-008-0027-x.
- [2] Grzegorz Bancerek. Curried and uncurried functions. *Formalized Mathematics*, 1(3):537–541, 1990.
- [3] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(1):107–114, 1990.
- [5] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Kornilowicz, Roman Matuszewski, Adam Naumowicz, Karol Pąk, and Josef Urban. Mizar: State-of-the-art and beyond. In Manfred Kerber, Jacques Carette, Cezary Kaliszyk, Florian Rabe, and Volker Sorge, editors, *Intelligent Computer Mathematics*, volume 9150 of *Lecture Notes in Computer Science*, pages 261–279. Springer International Publishing, 2015. ISBN 978-3-319-20614-1. doi:10.1007/978-3-319-20615-8\_17.
- [6] Heinz Bauer. *Measure and Integration Theory*. Walter de Gruyter Inc.
- [7] Józef Białas. The  $\sigma$ -additive measure theory. *Formalized Mathematics*, 2(2):263–270, 1991.

- [8] Józef Białaś. Series of positive real numbers. Measure theory. *Formalized Mathematics*, 2(1):173–183, 1991.
- [9] Vladimir Igorevich Bogachev and Maria Aparecida Soares Ruas. *Measure theory*, volume 1. Springer, 2007.
- [10] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [11] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [12] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(1):245–254, 1990.
- [13] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [14] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [15] Noboru Endou. Construction of measure from semialgebra of sets. *Formalized Mathematics*, 23(4):309–323, 2015. doi:10.1515/forma-2015-0025.
- [16] Noboru Endou and Yasunari Shidama. Integral of measurable function. *Formalized Mathematics*, 14(2):53–70, 2006. doi:10.2478/v10037-006-0008-x.
- [17] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Definitions and basic properties of measurable functions. *Formalized Mathematics*, 9(3):495–500, 2001.
- [18] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. The measurability of extended real valued functions. *Formalized Mathematics*, 9(3):525–529, 2001.
- [19] Noboru Endou, Keiko Narita, and Yasunari Shidama. The Lebesgue monotone convergence theorem. *Formalized Mathematics*, 16(2):167–175, 2008. doi:10.2478/v10037-008-0023-1.
- [20] Noboru Endou, Hiroyuki Okazaki, and Yasunari Shidama. Hopf extension theorem of measure. *Formalized Mathematics*, 17(2):157–162, 2009. doi:10.2478/v10037-009-0018-6.
- [21] Gerald B. Folland. *Real Analysis: Modern Techniques and Their Applications*. Wiley, 2 edition, 1999.
- [22] P. R. Halmos. *Measure Theory*. Springer-Verlag, 1974.
- [23] Andrzej Nędzusiak.  $\sigma$ -fields and probability. *Formalized Mathematics*, 1(2):401–407, 1990.
- [24] Beata Perkowska. Functional sequence from a domain to a domain. *Formalized Mathematics*, 3(1):17–21, 1992.
- [25] M.M. Rao. *Measure Theory and Integration*. Marcel Dekker, 2nd edition, 2004.
- [26] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(2):329–334, 1990.
- [27] Wojciech A. Trybulec. Non-contiguous substrings and one-to-one finite sequences. *Formalized Mathematics*, 1(3):569–573, 1990.
- [28] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.
- [29] Hiroshi Yamazaki, Noboru Endou, Yasunari Shidama, and Hiroyuki Okazaki. Inferior limit, superior limit and convergence of sequences of extended real numbers. *Formalized Mathematics*, 15(4):231–236, 2007. doi:10.2478/v10037-007-0026-3.

Received December 31, 2015

---