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Underlying Simple Graphs

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Summary. In this article the notion of the underlying simple graph of a graph (as defined in [8]) is formalized in the Mizar system [5], along with some convenient variants. The property of a graph to be without decorators (as introduced in [7]) is formalized as well to serve as the base of graph enumerations in the future.

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0. Introduction

In the Mizar Mathematical Library [2] there are several formalizations of graphs with a varying degree of generality, see [1], [6], [10], [8], [9]. The `GLIB_` series (starting with [8]) formalizes general digraphs (that is, digraphs with loops and parallel edges allowed) in Mizar [5] and provides a rich notation so that any digraph in Mizar can be seen as an undirected graph simply by ignoring the direction of the edges (although they are always there). In conclusion, there is no need for another formalization of undirected graphs, in contrast to how it is typically done in the literature (cf. [12], [3]), and the underlying (undirected) graph of a digraph (in the sense of [8]) is itself. For undirected graphs or digraphs possibly containing loops and multiple parallel edges, the underlying (simple) graph or digraph is derived by removing the loops and replacing each

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set of parallel edges with a single edge. That concept requires formalization and this article provides subgraph modes that respectively remove loops, (directed) parallel edges or both from a given (di)graph. "Much of graph theory is concerned with the study of simple graphs" [4, p. 3] which results in many books only studying simple graphs, even when graphs are more generally introduced in the respective book (for example [11]).

The rather extensive preliminaries contain many theorems that would fit well into earlier articles of the `GLIB` series, for example:

- The source and target of a directed edge in a graph are uniquely determined.
- A walk in a graph is uniquely determined by its vertex and edge sequence.
- Adding vertices to a graph doesn't change adjacencies.

The next section introduces plain graphs. Graphs, as defined in [8], can arbitrarily be expanded with decorators as done in [7]. Therefore for any non empty set S the set containing all graphs with vertex and edge sets contained in S does not exist because of possible decorators, even if S only contains a single element. A graph is called `plain` if it does not contain additional decorators, and then the set of all plain graphs with vertex and edge sets contained in S can be constructed, which will be needed for graph enumeration at a later point in time.

In the section after that the set of all loops of a graph is introduced as well as a graph operator removing all loops from a given graph as a special case of removing edges.

At the start of the following section, two equivalence relations are defined on the edge set, where two edges are equivalent iff they are (directed) parallel. Then modes are introduced to pick one edge out of each set of (directed) parallel edges. Using such representative edge selections, the graphs with parallel edges removed can be defined as induced subgraphs. While the directed and undirected variants are formalized along each other, there are also some theorems focusing on how they interact with each other.

This trend is continued in the last section, where the underlying simple graphs are introduced as induced subgraphs on the representative edge selections with the loops removed. Naturally, these subgraphs can also be constructed by removing loops and then parallel edges from a graph or vice versa.

1. Preliminaries

Now we state the propositions:

(1) Let us consider sets X, Y. If $Y \subseteq X$, then $X \setminus (X \setminus Y) = Y$.

(2) Let us consider a binary relation R, and a set X. Then

(i) $(R \restriction X)^{\smile} = X \upharpoonleft R^{\smile}$, and

(ii) $(X \upharpoonleft R)^{\smile} = R^{\smile} \restriction X$.

Let us consider a function f and a set Y. Now we state the propositions:

(3) $\mathrm{dom}(Y \upharpoonleft f) = f^{-1}(Y)$.

Proof: For every object x, $x \in \mathrm{dom}(Y \upharpoonleft f)$ iff $x \in f^{-1}(Y)$. $\square$

(4) $Y \upharpoonleft f = f \restriction \mathrm{dom}(Y \upharpoonleft f)$. The theorem is a consequence of (3).

(5) Let us consider a one-to-one function f, and a set X. Then

(i) $(f \restriction X)^{-1} = X \upharpoonleft f^{-1}$, and

(ii) $(X \upharpoonleft f)^{-1} = f^{-1} \restriction X$.

The theorem is a consequence of (2).

(6) Let us consider a graph G, and objects e, x_1, y_1, x_2, y_2. Suppose e joins x_1 to y_1 in G and e joins x_2 to y_2 in G. Then

(i) $x_1 = x_2$, and

(ii) $y_1 = y_2$.

Let G be a trivial graph. Let us observe that the vertices of G is trivial and every graph which is trivial and non-directed-multi is also non-multi.

Let G be a trivial, non-directed-multi graph. Let us observe that the edges of G is trivial.

Now we state the propositions:

(7) Let us consider a graph G, sets X, Y, and objects e, x, y. Suppose e joins x to y in G and $x \in X$ and $y \in Y$. Then e joins a vertex from X to a vertex from Y in G.

(8) Let us consider a trivial graph G, and a graph H. Suppose the vertices of $H \subseteq$ the vertices of G and the edges of $H \subseteq$ the edges of G. Then H is trivial and subgraph of G.

(9) Let us consider a graph G. Then $G \approx G \restriction$(the graph selectors).

Let us consider a graph G, sets X, Y, and an object e. Now we state the propositions:

(10) e joins a vertex from X and a vertex from Y in G if and only if e joins a vertex from Y and a vertex from X in G.

(11) e joins a vertex from X and a vertex from Y in G if and only if e joins a vertex from X to a vertex from Y in G or e joins a vertex from Y to a vertex from X in G.

Let us consider a graph G and objects e, v, w. Now we state the propositions:

(12) If e joins a vertex from $\{v\}$ and a vertex from $\{w\}$ in G, then e joins v and w in G.

(13) If e joins a vertex from $\{v\}$ to a vertex from $\{w\}$ in G, then e joins v to w in G.

(14) Let us consider a graph G, and objects v, w. Suppose $v \neq w$. Then

(i) G.edgesDBetween($\{v\}, \{w\}$) misses G.edgesDBetween($\{w\}, \{v\}$), and

(ii) G.edgesBetween($\{v\}, \{w\}$) $=$ G.edgesDBetween($\{v\}, \{w\}$)$\cup$ G.edgesDBetween($\{w\}, \{v\}$).

The theorem is a consequence of (11).

(15) Let us consider a graph G, and a set X. Then G.edgesBetween(X, X) $=$ G.edgesDBetween(X, X). The theorem is a consequence of (11).

(16) Let us consider a graph G, and sets X, Y. Then G.edgesBetween(X, Y) $=$ G.edgesBetween(Y, X). The theorem is a consequence of (10).

Let us consider a graph G. Now we state the propositions:

(17) G is loopless if and only if for every object v, there exists no object e such that e joins v to v in G.

PROOF: For every object v, there exists no object e such that e joins v and v in G. $\square$

(18) G is loopless if and only if for every object v, there exists no object e such that e joins a vertex from $\{v\}$ and a vertex from $\{v\}$ in G.

PROOF: For every object v, there exists no object e such that e joins v and v in G. $\square$

(19) G is loopless if and only if for every object v, there exists no object e such that e joins a vertex from $\{v\}$ to a vertex from $\{v\}$ in G. The theorem is a consequence of (11) and (18).

(20) G is loopless if and only if for every object v, G.edgesBetween($\{v\}, \{v\}$) $=$ $\emptyset$. The theorem is a consequence of (18).

(21) G is loopless if and only if for every object v, G.edgesDBetween($\{v\}, \{v\}$) $=$ $\emptyset$. The theorem is a consequence of (19).

Let G be a loopless graph and v be an object. One can verify that G.edgesBetween($\{v\}, \{v\}$) is empty and G.edgesDBetween($\{v\}, \{v\}$) is empty.

(22) Let us consider a graph G. Then G is non-multi if and only if for every objects v, w, G.edgesBetween($\{v\}, \{w\}$) is trivial. The theorem is a consequence of (12).

Let G be a non-multi graph and v, w be objects. One can verify that G.edgesBetween($\{v\}, \{w\}$) is trivial. Now we state the proposition:

(23) Let us consider a graph G. Then G is non-directed-multi if and only if for every objects v, w, G.edgesDBetween($\{v\}, \{w\}$) is trivial. The theorem is a consequence of (13) and (7).

Let G be a non-directed-multi graph and v, w be objects. One can check that G.edgesDBetween($\{v\}, \{w\}$) is trivial.

Let G be a non trivial graph. Let us note that every subgraph of G which is spanning is also non trivial.

Let G be a graph. One can check that every vertex of G which is isolated is also non endvertex.

Let us consider a graph G and a vertex v of G. Now we state the propositions:

(24) (G.walkOf(v)).edgeSeq() $= \varepsilon_{\alpha}$, where α is the edges of G.

(25) (G.walkOf(v)).edges() $= \emptyset$. The theorem is a consequence of (24).

Let G be a graph and W be a trivial walk of G. Note that W.edges() is empty and trivial.

Let W be a walk of G. Note that W.vertices() is non empty.

Now we state the propositions:

(26) Let us consider graphs G_1, G_2, a walk W_1 of G_1, and a walk W_2 of G_2. Suppose W_1.vertexSeq() $=$ W_2.vertexSeq() and W_1.edgeSeq() $=$ W_2.edgeSeq(). Then $W_1 = W_2$.

Proof: For every natural number n such that $1 \leqslant n \leqslant \operatorname{len} W_1$ holds $W_1(n) = W_2(n)$. $\square$

(27) Let us consider a graph G, a finite sequence p of elements of the vertices of G, and a finite sequence q of elements of the edges of G. Suppose $\operatorname{len} p = 1 + \operatorname{len} q$ and for every element n of $\mathbb{N}$ such that $1 \leqslant n$ and $n + 1 \leqslant \operatorname{len} p$ holds $q(n)$ joins $p(n)$ and $p(n + 1)$ in G. Then there exists a walk W of G such that

(i) W.vertexSeq() $= p$, and

(ii) W.edgeSeq() $= q$.

Proof: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a natural number m such that $m = \$_1$ and if m is odd, then $\$_2 = p(m{+}1 \operatorname{div} 2)$ and if m is even, then $\$_2 = q(m \operatorname{div} 2)$. For every natural number k such that $k \in \operatorname{Seg}(\operatorname{len} p{+}\operatorname{len} q)$ there exists an element x of (the vertices of G)$\cup$(the edges of G) such that $\mathcal{P}[k, x]$. Consider W being a finite sequence of elements of (the vertices of

$G) \cup$ (the edges of G) such that $\operatorname{dom} W = \operatorname{Seg}(\operatorname{len} p + \operatorname{len} q)$ and for every natural number k such that $k \in \operatorname{Seg}(\operatorname{len} p + \operatorname{len} q)$ holds $\mathcal{P}[k, W(k)]$. $W(1) \in$ the vertices of G. For every odd element n of $\mathbb{N}$ such that $n < \operatorname{len} W$ holds $W(n+1)$ joins $W(n)$ and $W(n+2)$ in G. For every natural number k such that $1 \leqslant k \leqslant \operatorname{len} p$ holds $p(k) = (W.\mathrm{vertexSeq}())(k)$. For every natural number k such that $1 \leqslant k \leqslant \operatorname{len} q$ holds $q(k) = (W.\mathrm{edgeSeq}())(k)$. □

(28) Let us consider a graph G, and a walk W of G. Then $\operatorname{len}(W.\mathrm{vertexSeq}()) = W.\mathrm{length}() + 1$.

(29) Let us consider graphs G_1, G_2, a walk W_1 of G_1, a walk W_2 of G_2, and an odd natural number n. If $W_1.\mathrm{vertexSeq}() = W_2.\mathrm{vertexSeq}()$, then $W_1(n) = W_2(n)$.

Let us consider graphs G_1, G_2, a walk W_1 of G_1, and a walk W_2 of G_2. Now we state the propositions:

(30) Suppose $W_1.\mathrm{vertexSeq}() = W_2.\mathrm{vertexSeq}()$. Then

(i) $\operatorname{len} W_1 = \operatorname{len} W_2$, and

(ii) $W_1.\mathrm{length}() = W_2.\mathrm{length}()$, and

(iii) $W_1.\mathrm{first}() = W_2.\mathrm{first}()$, and

(iv) $W_1.\mathrm{last}() = W_2.\mathrm{last}()$, and

(v) W_2 is walk from $W_1.\mathrm{first}()$ to $W_1.\mathrm{last}()$.

The theorem is a consequence of (29).

(31) If $W_1.\mathrm{vertexSeq}() = W_2.\mathrm{vertexSeq}()$, then if W_1 is not trivial, then W_2 is not trivial and if W_1 is closed, then W_2 is closed. The theorem is a consequence of (30).

(32) Suppose $W_1.\mathrm{vertexSeq}() = W_2.\mathrm{vertexSeq}()$ and $\operatorname{len} W_1 \neq 5$. Then

(i) if W_1 is path-like, then W_2 is path-like, and

(ii) if W_1 is cycle-like, then W_2 is cycle-like.

Proof: If W_1 is path-like, then W_2 is path-like. □

The scheme *IndWalk* deals with a graph $\mathcal{G}$ and a unary predicate $\mathcal{P}$ and states that

(Sch. 1) For every walk W of $\mathcal{G}$, $\mathcal{P}[W]$

provided

- for every trivial walk W of $\mathcal{G}$, $\mathcal{P}[W]$ and
- for every walk W of $\mathcal{G}$ and for every object e such that $e \in W.\mathrm{last}().\mathrm{edgesInOut}()$ and $\mathcal{P}[W]$ holds $\mathcal{P}[W.\mathrm{addEdge}(e)]$.

The scheme *IndDWalk* deals with a graph $\mathcal{G}$ and a unary predicate $\mathcal{P}$ and states that

(Sch. 2) For every dwalk W of $\mathcal{G}$, $\mathcal{P}[W]$

provided

- for every trivial dwalk W of $\mathcal{G}$, $\mathcal{P}[W]$ and
- for every dwalk W of $\mathcal{G}$ and for every object e such that $e \in W$.last().edgesOut() and $\mathcal{P}[W]$ holds $\mathcal{P}[W$.addEdge(e)].

Now we state the propositions:

(33) Let us consider a graph G_1, a subset E of the edges of G_1, and a subgraph G_2 of G_1 induced by the vertices of G_1 and E. If G_2 is connected, then G_1 is connected.

(34) Let us consider a graph G_1, a set E, and a subgraph G_2 of G_1 with edges E removed. If G_2 is connected, then G_1 is connected.

Let G_1 be a non connected graph and E be a set. One can check that every subgraph of G_1 with edges E removed is non connected.

(35) Let us consider a graph G_1, and a subgraph G_2 of G_1. Suppose for every walk W_1 of G_1, there exists a walk W_2 of G_2 such that W_2 is walk from W_1.first() to W_1.last(). Let us consider a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then G_1.reachableFrom(v_1) $=$ G_2.reachableFrom(v_2).

(36) Let us consider a graph G_1, and a subgraph G_2 of G_1. Suppose for every walk W_1 of G_1, there exists a walk W_2 of G_2 such that W_2 is walk from W_1.first() to W_1.last(). If G_1 is connected, then G_2 is connected.

Let us consider a graph G_1 and a spanning subgraph G_2 of G_1. Now we state the propositions:

(37) Suppose for every vertex v_1 of G_1 and for every vertex v_2 of G_2 such that $v_1 = v_2$ holds G_1.reachableFrom(v_1) $=$ G_2.reachableFrom(v_2). Then G_1.componentSet() $=$ G_2.componentSet().

(38) Suppose for every vertex v_1 of G_1 and for every vertex v_2 of G_2 such that $v_1 = v_2$ holds G_1.reachableFrom(v_1) $=$ G_2.reachableFrom(v_2). Then G_1.numComponents() $=$ G_2.numComponents(). The theorem is a consequence of (37).

(39) Let us consider a graph G. Then G is loopless if and only if for every vertex v of G, v and v are not adjacent.

Let G be a non complete graph. One can check that every subgraph of G which is spanning is also non complete.

Now we state the propositions:

(40) Let us consider graphs G_2, G_3, and a supergraph G_1 of G_3. If $G_1 \approx G_2$, then G_2 is a supergraph of G_3.

(41) Let us consider a graph G_2, a set V, a supergraph G_1 of G_2 extended by the vertices from V, sets x, y, and an object e. Then

(i) e joins x and y in G_1 iff e joins x and y in G_2, and

(ii) e joins x to y in G_1 iff e joins x to y in G_2, and

(iii) e joins a vertex from x and a vertex from y in G_1 iff e joins a vertex from x and a vertex from y in G_2, and

(iv) e joins a vertex from x to a vertex from y in G_1 iff e joins a vertex from x to a vertex from y in G_2.

(42) Let us consider graphs G_1, G_2. Suppose $G_1 \approx G_2$. Then G_2 is a graph given by reversing directions of the edges $\emptyset$ of G_1.

(43) Every graph is a graph given by reversing directions of the edges $\emptyset$ of G.

2. Plain Graphs

Let G be a graph. We say that G is plain if and only if

(Def. 1) $\operatorname{dom} G =$ the graph selectors.

Note that $G \restriction$(the graph selectors) is plain.

Let V be a non empty set, E be a set, and S, T be functions from E into V. Let us observe that createGraph(V, E, S, T) is plain.

Let G be a graph and X be a set. Note that G.set(WeightSelector$, X)$ is non plain and G.set(ELabelSelector$, X)$ is non plain and G.set(VLabelSelector$, X)$ is non plain and there exists a graph which is plain.

Now we state the proposition:

(44) Let us consider plain graphs G_1, G_2. If $G_1 \approx G_2$, then $G_1 = G_2$.

Let G be a graph. Note that there exists a subgraph of G which is plain.

Let V be a set. One can check that there exists a subgraph of G with vertices V removed which is plain.

Let E be a set. Let us note that there exists a subgraph of G induced by V and E which is plain and there exists a subgraph of G with edges E removed which is plain and there exists a graph given by reversing directions of the edges E of G which is plain.

Let v be a set. One can verify that there exists a subgraph of G with vertex v removed which is plain.

Let e be a set. One can verify that there exists a subgraph of G with edge e removed which is plain and there exists a supergraph of G which is plain.

Let V be a set. Let us note that there exists a supergraph of G extended by the vertices from V which is plain.

Let v, e, w be objects. One can check that there exists a supergraph of G extended by e between vertices v and w which is plain and there exists a supergraph of G extended by v, w and e between them which is plain.

Let v be an object and V be a set. Let us note that there exists a supergraph of G extended by vertex v and edges from V of G to v which is plain and there exists a supergraph of G extended by vertex v and edges from v to V of G which is plain and there exists a supergraph of G extended by vertex v and edges between v and V of G which is plain.

3. Graphs with Loops Removed

Let G be a graph. The functor G.loops() yielding a subset of the edges of G is defined by

(Def. 2) for every object e, $e \in it$ iff there exists an object v such that e joins v and v in G.

Now we state the propositions:

(45) Let us consider a graph G, and an object e. Then $e \in G$.loops() if and only if there exists an object v such that e joins v to v in G.

(46) Let us consider a graph G, and objects e, v, w. If e joins v and w in G and $v \neq w$, then $e \notin G$.loops().

(47) Let us consider a graph G. Then G is loopless if and only if G.loops() $= \emptyset$.

Let G be a loopless graph. Let us observe that G.loops() is empty.

Let G be a non loopless graph. Let us observe that G.loops() is non empty.

Now we state the propositions:

(48) Let us consider a graph G_1, and a subgraph G_2 of G_1. Then G_2.loops() $\subseteq$ G_1.loops(). The theorem is a consequence of (45).

(49) Let us consider a graph G_2, and a supergraph G_1 of G_2. Then G_2.loops() $\subseteq$ G_1.loops(). The theorem is a consequence of (48).

(50) Let us consider graphs G_1, G_2. If $G_1 \approx G_2$, then G_1.loops() $=$ G_2.loops(). The theorem is a consequence of (48).

(51) Let us consider a graph G_1, a set E, and a graph G_2 given by reversing directions of the edges E of G_1. Then G_1.loops() $=$ G_2.loops().

(52) Let us consider a graph G_2, a set V, and a supergraph G_1 of G_2 extended by the vertices from V. Then G_1.loops() $=$ G_2.loops(). The theorem is a consequence of (41) and (49).

(53) Let us consider a graph G_2, objects v_1, e, v_2, and a supergraph G_1 of G_2 extended by e between vertices v_1 and v_2. If $v_1 \neq v_2$, then G_1.loops() = G_2.loops(). The theorem is a consequence of (50) and (49).

(54) Let us consider a graph G_2, a vertex v of G_2, an object e, and a supergraph G_1 of G_2 extended by e between vertices v and v. Suppose $e \notin$ the edges of G_2. Then G_1.loops() = G_2.loops() $\cup \{e\}$. The theorem is a consequence of (45) and (49).

(55) Let us consider a graph G_2, objects v_1, e, v_2, and a supergraph G_1 of G_2 extended by v_1, v_2 and e between them. Then G_1.loops() = G_2.loops(). The theorem is a consequence of (49) and (50).

(56) Let us consider a graph G_2, an object v, a set V, and a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2. Then G_1.loops() = G_2.loops(). The theorem is a consequence of (49) and (50).

(57) Let us consider a graph G, and a path P of G. Then

  (i) P.edges() misses G.loops(), or

  (ii) there exist objects v, e such that e joins v and v in G and $P = G$.walkOf(v, e, v).

Let G be a graph. A subgraph of G with loops removed is a subgraph of G with edges G.loops() removed. Now we state the proposition:

(58) Let us consider a loopless graph G_1, and a graph G_2. Then $G_1 \approx G_2$ if and only if G_2 is a subgraph of G_1 with loops removed.

Let us consider graphs G_1, G_2 and a subgraph G_3 of G_1 with loops removed.

(59) $G_2 \approx G_3$ if and only if G_2 is a subgraph of G_1 with loops removed.

(60) If $G_1 \approx G_2$, then G_3 is a subgraph of G_2 with loops removed. The theorem is a consequence of (50).

Let G be a graph. Observe that every subgraph of G with loops removed is loopless and there exists a subgraph of G with loops removed which is plain.

Let G be a non-multi graph. Observe that every subgraph of G with loops removed is simple.

Let G be a non-directed-multi graph. One can check that every subgraph of G with loops removed is directed-simple.

Let G be a complete graph. Observe that every subgraph of G with loops removed is complete.

Now we state the propositions:

(61) Let us consider a graph G_1, a subgraph G_2 of G_1 with loops removed, and a walk W_1 of G_1. Then there exists a walk W_2 of G_2 such that W_2 is walk from W_1.first() to W_1.last(). The theorem is a consequence of (57).

(62) Let us consider a graph G_1, a subgraph G_2 of G_1 with loops removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then G_1.reachableFrom$(v_1) = G_2$.reachableFrom(v_2). The theorem is a consequence of (61) and (35).

Let G be a connected graph. Observe that every subgraph of G with loops removed is connected. Let G be a non connected graph. Observe that every subgraph of G with loops removed is non connected. Let us consider a graph G_1 and a subgraph G_2 of G_1 with loops removed. Now we state the propositions:

(63) G_1.componentSet() $= G_2$.componentSet(). The theorem is a consequence of (62) and (37).

(64) G_1.numComponents() $= G_2$.numComponents(). The theorem is a consequence of (62) and (38).

(65) G_1 is chordal if and only if G_2 is chordal. The theorem is a consequence of (46) and (57).

Let G be a chordal graph. Let us observe that every subgraph of G with loops removed is chordal. Now we state the proposition:

(66) Let us consider a graph G_1, a set v, a subgraph G_2 of G_1 with loops removed, and a subgraph G_3 of G_1 with vertex v removed. Then every subgraph of G_2 with vertex v removed is a subgraph of G_3 with loops removed. The theorem is a consequence of (1), (48), (59), and (60).

Let us consider a graph G_1, a subgraph G_2 of G_1 with loops removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. Now we state the propositions:

(67) If $v_1 = v_2$, then v_1 is cut-vertex iff v_2 is cut-vertex. The theorem is a consequence of (66) and (64).

(68) If $v_1 = v_2$ and v_1 is endvertex, then v_2 is endvertex. The theorem is a consequence of (46).

4. Graphs with Parallel Edges Removed

Let G be a graph. The functors: EdgeParEqRel(G) and DEdgeParEqRel(G) yielding equivalence relations of the edges of G are defined by conditions

(Def. 3) for all objects e_1, e_2, $\langle e_1, e_2\rangle \in$ EdgeParEqRel(G) iff there exist objects v_1, v_2 such that e_1 joins v_1 and v_2 in G and e_2 joins v_1 and v_2 in G,

(Def. 4) for all objects e_1, e_2, $\langle e_1, e_2\rangle \in$ DEdgeParEqRel(G) iff there exist objects v_1, v_2 such that e_1 joins v_1 to v_2 in G and e_2 joins v_1 to v_2 in G,

respectively.

Let us consider a graph G. Now we state the propositions:

(69) DEdgeParEqRel$(G) \subseteq$ EdgeParEqRel(G).

(70) G is non-multi if and only if $\mathrm{EdgeParEqRel}(G) = \mathrm{id}_\alpha$, where α is the edges of G.

(71) G is non-directed-multi if and only if $\mathrm{DEdgeParEqRel}(G) = \mathrm{id}_\alpha$, where α is the edges of G.

Let G be an edgeless graph. One can verify that $\mathrm{EdgeParEqRel}(G)$ is empty and $\mathrm{DEdgeParEqRel}(G)$ is empty.

Let G be a non edgeless graph. Observe that $\mathrm{EdgeParEqRel}(G)$ is non empty and $\mathrm{DEdgeParEqRel}(G)$ is non empty.

Let G be a graph.

A representative selection of the parallel edges of G is a subset of the edges of G defined by

(Def. 5) for every objects v, w, e_0 such that e_0 joins v and w in G there exists an object e such that e joins v and w in G and $e \in it$ and for every object e' such that e' joins v and w in G and $e' \in it$ holds $e' = e$.

A representative selection of the directed-parallel edges of G is a subset of the edges of G defined by

(Def. 6) for every objects v, w, e_0 such that e_0 joins v to w in G there exists an object e such that e joins v to w in G and $e \in it$ and for every object e' such that e' joins v to w in G and $e' \in it$ holds $e' = e$.

Let G be an edgeless graph. Let us observe that every representative selection of the parallel edges of G is empty and every representative selection of the directed-parallel edges of G is empty.

Let G be a non edgeless graph. Let us observe that every representative selection of the parallel edges of G is non empty and every representative selection of the directed-parallel edges of G is non empty.

Now we state the propositions:

(72) Let us consider a graph G, and a representative selection of the directed-parallel edges E_1 of G. Then there exists a representative selection of the parallel edges E_2 of G such that $E_2 \subseteq E_1$.

PROOF: Set $A = \{\{e$, where e is an element of the edges of $G : e$ joins v_1 and v_2 in G and $e \in E_1\}$, where v_1, v_2 are vertices of $G :$ there exists an object e_0 such that e_0 joins v_1 and v_2 in $G\}$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a non empty set S such that $\$_1 = S$ and $\$_2 =$ the element of S. For every object x such that $x \in A$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\mathrm{dom}\, f = A$ and for every object x such that $x \in A$ holds $\mathcal{P}[x, f(x)]$. For every object e such that $e \in \mathrm{rng}\, f$ holds $e \in E_1$. Reconsider $E_2 = \mathrm{rng}\, f$ as a subset of the edges of G. For every objects v, w, e_0 such that e_0 joins v and w in G there exists an object e such that e joins v and w in G and $e \in E_2$ and for every object

e' such that e' joins v and w in G and $e' \in E_2$ holds $e' = e$. $\square$

(73) Let us consider a graph G, and a representative selection of the parallel edges E_2 of G. Then there exists a representative selection of the directed-parallel edges E_1 of G such that $E_2 \subseteq E_1$.

Proof: Set $A = \{\{e$, where e is an element of the edges of $G : e$ joins v_1 to v_2 in $G\}$, where v_1, v_2 are vertices of G : there exists an object e_0 such that e_0 joins v_1 to v_2 in G and for every object e_0 such that e_0 joins v_1 to v_2 in G holds $e_0 \notin E_2\}$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a non empty set S such that $\$_1 = S$ and $\$_2 =$ the element of S. For every object x such that $x \in A$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f = A$ and for every object x such that $x \in A$ holds $\mathcal{P}[x, f(x)]$. For every object e such that $e \in \operatorname{rng} f$ holds $e \in$ the edges of G. Reconsider $E_1 = E_2 \cup \operatorname{rng} f$ as a subset of the edges of G. For every objects v, w, e_0 such that e_0 joins v to w in G there exists an object e such that e joins v to w in G and $e \in E_1$ and for every object e' such that e' joins v to w in G and $e' \in E_1$ holds $e' = e$. $\square$

(74) Let us consider a non-multi graph G, and a representative selection of the parallel edges E of G. Then $E =$ the edges of G.

(75) Let us consider a graph G. Suppose there exists a representative selection of the parallel edges E of G such that $E =$ the edges of G. Then G is non-multi.

(76) Let us consider a non-directed-multi graph G, and a representative selection of the directed-parallel edges E of G. Then $E =$ the edges of G.

(77) Let us consider a graph G. Suppose there exists a representative selection of the directed-parallel edges E of G such that $E =$ the edges of G. Then G is non-directed-multi.

(78) Let us consider a graph G_1, a subgraph G_2 of G_1, and a representative selection of the parallel edges E of G_1. Suppose $E \subseteq$ the edges of G_2. Then E is a representative selection of the parallel edges of G_2.

(79) Let us consider a graph G_1, a subgraph G_2 of G_1, and a representative selection of the directed-parallel edges E of G_1. Suppose $E \subseteq$ the edges of G_2. Then E is a representative selection of the directed-parallel edges of G_2.

(80) Let us consider a graph G_1, a subgraph G_2 of G_1, and a representative selection of the parallel edges E_2 of G_2. Then there exists a representative selection of the parallel edges E_1 of G_1 such that $E_2 = E_1 \cap$ (the edges of G_2).

Proof: Set $A = \{\{e$, where e is an element of the edges of $G_1 : e$ joins v_1

and v_2 in $G_1\}$, where v_1, v_2 are vertices of G_1 : there exists an object e_0 such that e_0 joins v_1 and v_2 in G_1 and for every object e_0 such that e_0 joins v_1 and v_2 in G_1 holds $e_0 \notin E_2\}$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a non empty set S such that $\$_1 = S$ and $\$_2 =$ the element of S. For every object x such that $x \in A$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f = A$ and for every object x such that $x \in A$ holds $\mathcal{P}[x, f(x)]$. For every object e such that $e \in \operatorname{rng} f$ holds $e \in$ the edges of G_1. Reconsider $E_1 = E_2 \cup \operatorname{rng} f$ as a subset of the edges of G_1. For every objects v, w, e_0 such that e_0 joins v and w in G_1 there exists an object e such that e joins v and w in G_1 and $e \in E_1$ and for every object e' such that e' joins v and w in G_1 and $e' \in E_1$ holds $e' = e$. For every object x, $x \in E_2$ iff $x \in E_1$ and $x \in$ the edges of G_2. □

(81) Let us consider a graph G_1, a subgraph G_2 of G_1, and a representative selection of the directed-parallel edges E_2 of G_2. Then there exists a representative selection of the directed-parallel edges E_1 of G_1 such that $E_2 = E_1 \cap (\text{the edges of } G_2)$.

PROOF: Set $A = \{\{e$, where e is an element of the edges of G_1 : e joins v_1 to v_2 in $G_1\}$, where v_1, v_2 are vertices of G_1 : there exists an object e_0 such that e_0 joins v_1 to v_2 in G_1 and for every object e_0 such that e_0 joins v_1 to v_2 in G_1 holds $e_0 \notin E_2\}$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a non empty set S such that $\$_1 = S$ and $\$_2 =$ the element of S. For every object x such that $x \in A$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f = A$ and for every object x such that $x \in A$ holds $\mathcal{P}[x, f(x)]$. For every object e such that $e \in \operatorname{rng} f$ holds $e \in$ the edges of G_1. Reconsider $E_1 = E_2 \cup \operatorname{rng} f$ as a subset of the edges of G_1. For every objects v, w, e_0 such that e_0 joins v to w in G_1 there exists an object e such that e joins v to w in G_1 and $e \in E_1$ and for every object e' such that e' joins v to w in G_1 and $e' \in E_1$ holds $e' = e$. For every object x, $x \in E_2$ iff $x \in E_1$ and $x \in$ the edges of G_2. □

(82) Let us consider a graph G_1, a representative selection of the parallel edges E_1 of G_1, a subgraph G_2 of G_1 induced by the vertices of G_1 and E_1, and a representative selection of the parallel edges E_2 of G_2. Then $E_1 = E_2$.

PROOF: For every object e such that $e \in E_1$ holds $e \in E_2$. □

(83) Let us consider a graph G_1, a representative selection of the directed-parallel edges E_1 of G_1, a subgraph G_2 of G_1 induced by the vertices of G_1 and E_1, and a representative selection of the directed-parallel edges E_2 of G_2. Then $E_1 = E_2$.

PROOF: For every object e such that $e \in E_1$ holds $e \in E_2$. □

(84) Let us consider a graph G_1, a representative selection of the directed-parallel edges E_1 of G_1, a subgraph G_2 of G_1 induced by the vertices of G_1 and E_1, and a representative selection of the parallel edges E_2 of G_2. Then

(i) $E_2 \subseteq E_1$, and

(ii) E_2 is a representative selection of the parallel edges of G_1.

Let us consider a graph G and representative selections of the parallel edges E_1, E_2 of G. Now we state the propositions:

(85) There exists a one-to-one function f such that

(i) $\mathrm{dom}\, f = E_1$, and

(ii) $\mathrm{rng}\, f = E_2$, and

(iii) for every objects e, v, w such that $e \in E_1$ holds e joins v and w in G iff $f(e)$ joins v and w in G.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv \$_2 \in E_2$ and there exist objects v, w such that $\$_1$ joins v and w in G and $\$_2$ joins v and w in G. For every objects x, y_1, y_2 such that $x \in E_1$ and $\mathcal{P}[x, y_1]$ and $\mathcal{P}[x, y_2]$ holds $y_1 = y_2$. For every object x such that $x \in E_1$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\mathrm{dom}\, f = E_1$ and for every object x such that $x \in E_1$ holds $\mathcal{P}[x, f(x)]$. Consider v_0, w_0 being objects such that e joins v_0 and w_0 in G and $f(e)$ joins v_0 and w_0 in G. $\square$

(86) $\overline{\overline{E_1}} = \overline{\overline{E_2}}$. The theorem is a consequence of (85).

Let us consider a graph G and representative selections of the directed-parallel edges E_1, E_2 of G. Now we state the propositions:

(87) There exists a one-to-one function f such that

(i) $\mathrm{dom}\, f = E_1$, and

(ii) $\mathrm{rng}\, f = E_2$, and

(iii) for every objects e, v, w such that $e \in E_1$ holds e joins v to w in G iff $f(e)$ joins v to w in G.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv \$_2 \in E_2$ and there exist objects v, w such that $\$_1$ joins v to w in G and $\$_2$ joins v to w in G. For every objects x, y_1, y_2 such that $x \in E_1$ and $\mathcal{P}[x, y_1]$ and $\mathcal{P}[x, y_2]$ holds $y_1 = y_2$. For every object x such that $x \in E_1$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\mathrm{dom}\, f = E_1$ and for every object x such that $x \in E_1$ holds $\mathcal{P}[x, f(x)]$. Consider v_0, w_0 being objects such that e joins v_0 to w_0 in G and $f(e)$ joins v_0 to w_0 in G. $v_0 = v$ and $w_0 = w$. $\square$

(88) $\overline{\overline{E_1}} = \overline{\overline{E_2}}$. The theorem is a consequence of (87).

Let G be a graph.

A subgraph of G with parallel edges removed is a subgraph of G defined by

(Def. 7) there exists a representative selection of the parallel edges E of G such that *it* is a subgraph of G induced by the vertices of G and E.

A subgraph of G with directed-parallel edges removed is a subgraph of G defined by

(Def. 8) there exists a representative selection of the directed-parallel edges E of G such that *it* is a subgraph of G induced by the vertices of G and E.

Observe that every subgraph of G with parallel edges removed is spanning and non-multi and every subgraph of G with directed-parallel edges removed is spanning and non-directed-multi and there exists a subgraph of G with parallel edges removed which is plain and there exists a subgraph of G with directed-parallel edges removed which is plain.

Let G be a loopless graph. Let us observe that every subgraph of G with parallel edges removed is simple and every subgraph of G with directed-parallel edges removed is directed-simple.

Now we state the propositions:

(89) Let us consider a non-multi graph G_1, and a graph G_2. Then $G_1 \approx G_2$ if and only if G_2 is a subgraph of G_1 with parallel edges removed. The theorem is a consequence of (74).

(90) Let us consider a non-directed-multi graph G_1, and a graph G_2. Then $G_1 \approx G_2$ if and only if G_2 is a subgraph of G_1 with directed-parallel edges removed. The theorem is a consequence of (76).

(91) Let us consider graphs G_1, G_2, and a subgraph G_3 of G_1 with parallel edges removed. If $G_1 \approx G_2$, then G_3 is a subgraph of G_2 with parallel edges removed. The theorem is a consequence of (78).

(92) Let us consider graphs G_1, G_2, and a subgraph G_3 of G_1 with directed-parallel edges removed. Suppose $G_1 \approx G_2$. Then G_3 is a subgraph of G_2 with directed-parallel edges removed. The theorem is a consequence of (79).

(93) Let us consider graphs G_1, G_2, and a subgraph G_3 of G_1 with parallel edges removed. If $G_2 \approx G_3$, then G_2 is a subgraph of G_1 with parallel edges removed.

(94) Let us consider graphs G_1, G_2, and a subgraph G_3 of G_1 with directed-parallel edges removed. Suppose $G_2 \approx G_3$. Then G_2 is a subgraph of G_1 with directed-parallel edges removed.

Let us consider a graph G_1 and a subgraph G_2 of G_1 with directed-parallel edges removed. Now we state the propositions:

(95) Every subgraph of G_2 with parallel edges removed is a subgraph of G_1 with parallel edges removed. The theorem is a consequence of (84).

(96) There exists a subgraph G_3 of G_1 with parallel edges removed such that G_3 is a subgraph of G_2 with parallel edges removed. The theorem is a consequence of (72) and (78).

(97) Let us consider a graph G_1, and a subgraph G_3 of G_1 with parallel edges removed. Then there exists a subgraph G_2 of G_1 with directed-parallel edges removed such that G_3 is a subgraph of G_2 with parallel edges removed. The theorem is a consequence of (73) and (78).

Let G be a complete graph. Let us observe that every subgraph of G with parallel edges removed is complete and every subgraph of G with directed-parallel edges removed is complete.

Now we state the propositions:

(98) Let us consider a graph G_1, a subgraph G_2 of G_1 with parallel edges removed, and a walk W_1 of G_1. Then there exists a walk W_2 of G_2 such that W_1.vertexSeq() = W_2.vertexSeq().

PROOF: Define $\mathcal{P}$[walk of G_1] $\equiv$ there exists a walk W_2 of G_2 such that $\$_1$.vertexSeq() = W_2.vertexSeq(). For every trivial walk W of G_1, $\mathcal{P}[W]$. For every walk W of G_1 and for every object e such that $e \in W$.last().edgesInOut() and $\mathcal{P}[W]$ holds $\mathcal{P}[W$.addEdge$(e)]$. For every walk W_1 of G_1, $\mathcal{P}[W_1]$. $\square$

(99) Let us consider a graph G_1, a subgraph G_2 of G_1 with directed-parallel edges removed, and a walk W_1 of G_1. Then there exists a walk W_2 of G_2 such that W_1.vertexSeq() = W_2.vertexSeq(). The theorem is a consequence of (95) and (98).

(100) Let us consider a graph G_1, a subgraph G_2 of G_1 with parallel edges removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then G_1.reachableFrom(v_1) = G_2.reachableFrom(v_2). The theorem is a consequence of (35).

(101) Let us consider a graph G_1, a subgraph G_2 of G_1 with directed-parallel edges removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then G_1.reachableFrom(v_1) = G_2.reachableFrom(v_2). The theorem is a consequence of (35).

Let G be a connected graph. Note that every subgraph of G with parallel edges removed is connected and every subgraph of G with directed-parallel edges removed is connected.

Let G be a non connected graph. One can verify that every subgraph of G with parallel edges removed is non connected and every subgraph of G with directed-parallel edges removed is non connected.

Now we state the propositions:

(102) Let us consider a graph G_1, and a subgraph G_2 of G_1 with parallel edges removed. Then G_1.componentSet() $=$ G_2.componentSet(). The theorem is a consequence of (100) and (37).

(103) Let us consider a graph G_1, and a subgraph G_2 of G_1 with directed-parallel edges removed. Then G_1.componentSet() $=$ G_2.componentSet(). The theorem is a consequence of (101) and (37).

(104) Let us consider a graph G_1, and a subgraph G_2 of G_1 with parallel edges removed. Then G_1.numComponents() $=$ G_2.numComponents(). The theorem is a consequence of (100) and (38).

(105) Let us consider a graph G_1, and a subgraph G_2 of G_1 with directed-parallel edges removed. Then G_1.numComponents() $=$ G_2.numComponents(). The theorem is a consequence of (101) and (38).

(106) Let us consider a graph G_1, and a subgraph G_2 of G_1 with parallel edges removed. Then G_1 is chordal if and only if G_2 is chordal. The theorem is a consequence of (98), (30), (32), and (29).

(107) Let us consider a graph G_1, and a subgraph G_2 of G_1 with directed-parallel edges removed. Then G_1 is chordal if and only if G_2 is chordal. The theorem is a consequence of (95) and (106).

Let G be a chordal graph. Note that every subgraph of G with parallel edges removed is chordal and every subgraph of G with directed-parallel edges removed is chordal.

Now we state the propositions:

(108) Let us consider a graph G_1, a set v, a subgraph G_2 of G_1 with parallel edges removed, and a subgraph G_3 of G_1 with vertex v removed. Then every subgraph of G_2 with vertex v removed is a subgraph of G_3 with parallel edges removed. The theorem is a consequence of (93) and (91).

(109) Let us consider a graph G_1, a subgraph G_2 of G_1 with parallel edges removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then v_1 is cut-vertex iff v_2 is cut-vertex. The theorem is a consequence of (108) and (104).

(110) Let us consider a graph G_1, a subgraph G_2 of G_1 with directed-parallel edges removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then v_1 is cut-vertex iff v_2 is cut-vertex. The theorem is a consequence of (95) and (109).

(111) Let us consider a graph G_1, a subgraph G_2 of G_1 with parallel edges removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then v_1 is isolated iff v_2 is isolated.

PROOF: v_1.edgesInOut() $= \emptyset$. □

(112) Let us consider a graph G_1, a subgraph G_2 of G_1 with directed-parallel edges removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then v_1 is isolated iff v_2 is isolated. The theorem is a consequence of (95) and (111).

(113) Let us consider a graph G_1, a subgraph G_2 of G_1 with parallel edges removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$ and v_1 is endvertex, then v_2 is endvertex. The theorem is a consequence of (111).

(114) Let us consider a graph G_1, a subgraph G_2 of G_1 with directed-parallel edges removed, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$ and v_1 is endvertex, then v_2 is endvertex. The theorem is a consequence of (112).

Let G be a graph. A simple graph of G is a subgraph of G defined by

(Def. 9) there exists a representative selection of the parallel edges E of G such that *it* is a subgraph of G induced by the vertices of G and $E \setminus (G.\mathrm{loops}())$.

A directed-simple graph of G is a subgraph of G defined by

(Def. 10) there exists a representative selection of the directed-parallel edges E of G such that *it* is a subgraph of G induced by the vertices of G and $E \setminus (G.\mathrm{loops}())$.

Now we state the propositions:

(115) Let us consider a graph G_1, and a subgraph G_2 of G_1 with parallel edges removed. Then every subgraph of G_2 with loops removed is a simple graph of G_1. The theorem is a consequence of (48).

(116) Let us consider a graph G_1, and a subgraph G_2 of G_1 with directed-parallel edges removed. Then every subgraph of G_2 with loops removed is a directed-simple graph of G_1. The theorem is a consequence of (48).

Let us consider a graph G_1 and a subgraph G_2 of G_1 with loops removed. Now we state the propositions:

(117) Every subgraph of G_2 with parallel edges removed is a simple graph of G_1. The theorem is a consequence of (80).

(118) Every subgraph of G_2 with directed-parallel edges removed is a directed-simple graph of G_1. The theorem is a consequence of (81).

(119) Let us consider a graph G_1, and a simple graph G_3 of G_1. Then there exists a subgraph G_2 of G_1 with parallel edges removed such that G_3 is a subgraph of G_2 with loops removed.

PROOF: Consider E being a representative selection of the parallel edges of G_1 such that G_3 is a subgraph of G_1 induced by the vertices of G_1 and $E \setminus (G_1.\mathrm{loops}())$. Set $G_2 =$ the subgraph of G_1 induced by the vertices of

G_1 and E. For every object e, $e \in$ the edges of G_3 iff $e \in$ (the edges of G_2) $\setminus$ (G_2.loops()). $\square$

(120) Let us consider a graph G_1, and a directed-simple graph G_3 of G_1. Then there exists a subgraph G_2 of G_1 with directed-parallel edges removed such that G_3 is a subgraph of G_2 with loops removed.

PROOF: Consider E being a representative selection of the directed-parallel edges of G_1 such that G_3 is a subgraph of G_1 induced by the vertices of G_1 and $E \setminus$(G_1.loops()). Set $G_2 =$ the subgraph of G_1 induced by the vertices of G_1 and E. For every object e, $e \in$ the edges of G_3 iff $e \in$ (the edges of G_2) $\setminus$ (G_2.loops()). $\square$

Let us consider a graph G_1 and a subgraph G_2 of G_1 with loops removed. Now we state the propositions:

(121) Every simple graph of G_1 is a subgraph of G_2 with parallel edges removed.

(122) Every directed-simple graph of G_1 is a subgraph of G_2 with directed-parallel edges removed. The theorem is a consequence of (45) and (6).

Let us consider a loopless graph G_1 and a graph G_2. Now we state the propositions:

(123) G_2 is a simple graph of G_1 if and only if G_2 is a subgraph of G_1 with parallel edges removed.

(124) G_2 is a directed-simple graph of G_1 if and only if G_2 is a subgraph of G_1 with directed-parallel edges removed.

(125) Let us consider a non-multi graph G_1, and a graph G_2. Then G_2 is a simple graph of G_1 if and only if G_2 is a subgraph of G_1 with loops removed. The theorem is a consequence of (74).

(126) Let us consider a non-directed-multi graph G_1, and a graph G_2. Then G_2 is a directed-simple graph of G_1 if and only if G_2 is a subgraph of G_1 with loops removed. The theorem is a consequence of (76).

Let G be a graph. Note that every simple graph of G is spanning, loopless, non-multi, and simple and every directed-simple graph of G is spanning, loopless, non-directed-multi, and directed-simple and there exists a simple graph of G which is plain and there exists a directed-simple graph of G which is plain.

Now we state the propositions:

(127) Let us consider a simple graph G_1, and a graph G_2. Then $G_1 \approx G_2$ if and only if G_2 is a simple graph of G_1. The theorem is a consequence of (74).

(128) Let us consider a directed-simple graph G_1, and a graph G_2. Then $G_1 \approx G_2$ if and only if G_2 is a directed-simple graph of G_1. The theorem is

a consequence of (76).

(129) Let us consider graphs G_1, G_2, and a simple graph G_3 of G_1. If $G_1 \approx G_2$, then G_3 is a simple graph of G_2. The theorem is a consequence of (50) and (78).

(130) Let us consider graphs G_1, G_2, and a directed-simple graph G_3 of G_1. If $G_1 \approx G_2$, then G_3 is a directed-simple graph of G_2. The theorem is a consequence of (50) and (79).

(131) Let us consider graphs G_1, G_2, and a simple graph G_3 of G_1. If $G_2 \approx G_3$, then G_2 is a simple graph of G_1.

(132) Let us consider graphs G_1, G_2, and a directed-simple graph G_3 of G_1. If $G_2 \approx G_3$, then G_2 is a directed-simple graph of G_1.

Let us consider a graph G_1 and a directed-simple graph G_2 of G_1. Now we state the propositions:

(133) Every simple graph of G_2 is a simple graph of G_1. The theorem is a consequence of (122), (123), (95), and (117).

(134) There exists a simple graph G_3 of G_1 such that G_3 is a simple graph of G_2. The theorem is a consequence of (122), (96), (117), and (123).

(135) Let us consider a graph G_1, and a simple graph G_3 of G_1. Then there exists a directed-simple graph G_2 of G_1 such that G_3 is a simple graph of G_2. The theorem is a consequence of (121), (97), (118), and (123).

Let G be a complete graph. Observe that every simple graph of G is complete and every directed-simple graph of G is complete.

Now we state the propositions:

(136) Let us consider a graph G_1, a simple graph G_2 of G_1, and a walk W_1 of G_1. Then there exists a walk W_2 of G_2 such that W_2 is walk from W_1.first() to W_1.last(). The theorem is a consequence of (119) and (61).

(137) Let us consider a graph G_1, a directed-simple graph G_2 of G_1, and a walk W_1 of G_1. Then there exists a walk W_2 of G_2 such that W_2 is walk from W_1.first() to W_1.last(). The theorem is a consequence of (133) and (136).

(138) Let us consider a graph G_1, a simple graph G_2 of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then G_1.reachableFrom(v_1) = G_2.reachableFrom(v_2). The theorem is a consequence of (136) and (35).

(139) Let us consider a graph G_1, a directed-simple graph G_2 of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then G_1.reachableFrom(v_1) = G_2.reachableFrom(v_2). The theorem is a consequence of (137) and (35).

Let G be a connected graph. Observe that every simple graph of G is connected and every directed-simple graph of G is connected.

Let G be a non connected graph. One can verify that every simple graph of G is non connected and every directed-simple graph of G is non connected.

Now we state the propositions:

(140) Let us consider a graph G_1, and a simple graph G_2 of G_1. Then G_1.componentSet() $= G_2$.componentSet(). The theorem is a consequence of (138) and (37).

(141) Let us consider a graph G_1, and a directed-simple graph G_2 of G_1. Then G_1.componentSet() $= G_2$.componentSet(). The theorem is a consequence of (139) and (37).

(142) Let us consider a graph G_1, and a simple graph G_2 of G_1. Then G_1.numComponents() $= G_2$.numComponents(). The theorem is a consequence of (138) and (38).

(143) Let us consider a graph G_1, and a directed-simple graph G_2 of G_1. Then G_1.numComponents() $= G_2$.numComponents(). The theorem is a consequence of (139) and (38).

(144) Let us consider a graph G_1, and a simple graph G_2 of G_1. Then G_1 is chordal if and only if G_2 is chordal. The theorem is a consequence of (119), (65), and (106).

(145) Let us consider a graph G_1, and a directed-simple graph G_2 of G_1. Then G_1 is chordal if and only if G_2 is chordal. The theorem is a consequence of (120), (65), and (107).

Let G be a chordal graph. One can check that every simple graph of G is chordal and every directed-simple graph of G is chordal.

Now we state the propositions:

(146) Let us consider a graph G_1, a simple graph G_2 of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then v_1 is cut-vertex iff v_2 is cut-vertex. The theorem is a consequence of (119), (67), and (109).

(147) Let us consider a graph G_1, a directed-simple graph G_2 of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then v_1 is cut-vertex iff v_2 is cut-vertex. The theorem is a consequence of (120), (67), and (110).

(148) Let us consider a loopless graph G_1, a simple graph G_2 of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then v_1 is isolated iff v_2 is isolated. The theorem is a consequence of (119), (58), and (111).

(149) Let us consider a loopless graph G_1, a directed-simple graph G_2 of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then v_1 is isolated iff v_2 is isolated. The theorem is a consequence of (120), (58), and (112).

(150) Let us consider a graph G_1, a simple graph G_2 of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$ and v_1 is endvertex, then v_2 is endvertex.

The theorem is a consequence of (119), (113), and (68).

(151) Let us consider a graph G_1, a directed-simple graph G_2 of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$ and v_1 is endvertex, then v_2 is endvertex. The theorem is a consequence of (120), (114), and (68).

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