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About Vertex Mappings

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Summary. In [6] partial graph mappings were formalized in the Mizar system [3]. Such mappings map some vertices and edges of a graph to another while preserving adjacency. While this general approach is appropriate for the general form of (multidi)graphs as introduced in [7], a more specialized version for graphs without parallel edges seems convenient. As such, partial vertex mappings preserving adjacency between the mapped verticed are formalized here.

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0. Introduction

This article is a brief introduction to partial vertex mappings in Mizar [2]. As discussed in the introduction of [6] almost no graph theory book discusses graph homomorphisms in a scope as general as it was done in [5] and [6]. Most of the time, graph homomorphisms are only discussed in the form of vertex mappings, often only in the context of simple graphs. But of course that choice is not without reason and in many cases considering vertex mappings is enough, which is especially useful since one does not need to think about an edge mapping then. Given that the graph definitions change slightly between different authors, a quick overview of the formalized notation seems in order.

A *partial vertex mapping* f between two graphs G_1, G_2 is a partial function of their vertex sets $V(G_1)$, $V(G_2)$ with the additional property that if vertices

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$v, w \in \operatorname{dom} f$ are adjacent in G_1, then their images $f(v)$, $f(w)$ are adjacent in G_2. The properties of f to be *total* (or a homomorphism), *one-to-one* (or injective) and *onto* (or surjective) have the usual meaning for f as a partial function. f is *continuous* if for any $v, w \in \operatorname{dom} f$ such that $f(v)$ and $f(w)$ are adjacent, v and w are adjacent as well. f is an *isomorphism* if it is total, one-to-one, onto and the cardinality of edges between to vertices v and w of G_1 is the same as the cardinality of the edges between $f(v)$ and $f(w)$. Corresponding attributes for directed vertex mappings are given as well in this article.

The attribute *continuous* is the generalization for not necessarily simple graphs of the *continuous* of [5]. The *isomorphism* attribute was inspired by [1]. It is shown that for graphs G_1, G_2 without multiple edges that a total bijective and continuous vertex mapping f between them is already an isomorphism, just like a graph isomorphism is usually described (cf. [4], [8], [5]). This article does not go into depth like [6], but the inverse and composition of partial vertex mappings are covered.

A partial graph mapping does not always induce a partial vertex mapping (since any subset of the set of edges of G_1 can be mapped) and a partial vertex mapping can give rise to several partial graph mappings. In the second part of this article it is shown when the induced partial vertex mapping exists and when the induced partial graph mapping is unique. Furthermore it is formally stated that for two graphs without parallel edges there exists a graph mapping that is an isomorphism iff there exists a vertex mapping that is an isomorphism.

1. Vertex Mappings

Let G_1, G_2 be graphs.

A partial vertex mapping from G_1 to G_2 is a partial function from the vertices of G_1 to the vertices of G_2 defined by

(Def. 1) for every vertices v, w of G_1 such that v, $w \in \operatorname{dom} it$ and v and w are adjacent holds $it_{/v}$ and $it_{/w}$ are adjacent.

Now we state the proposition:

(1) Let us consider graphs G_1, G_2, and a partial function f from the vertices of G_1 to the vertices of G_2. Then f is a partial vertex mapping from G_1 to G_2 if and only if for every objects v, w, e such that v, $w \in \operatorname{dom} f$ and e joins v and w in G_1 there exists an object $\tilde{e}$ such that $\tilde{e}$ joins $f(v)$ and $f(w)$ in G_2.

Let G_1, G_2 be graphs and f be a partial vertex mapping from G_1 to G_2. We say that f is directed if and only if

(Def. 2) for every objects v, w, e such that $v, w \in \operatorname{dom} f$ and e joins v to w in G_1 there exists an object $\tilde{e}$ such that $\tilde{e}$ joins $f(v)$ to $f(w)$ in G_2.

We say that f is continuous if and only if

(Def. 3) for every vertices v, w of G_1 such that $v, w \in \operatorname{dom} f$ and $f_{/v}$ and $f_{/w}$ are adjacent holds v and w are adjacent.

We say that f is directed-continuous if and only if

(Def. 4) for every objects v, w, $\tilde{e}$ such that $v, w \in \operatorname{dom} f$ and $\tilde{e}$ joins $f(v)$ to $f(w)$ in G_2 there exists an object e such that e joins v to w in G_1.

Let us consider graphs G_1, G_2 and a partial vertex mapping f from G_1 to G_2. Now we state the propositions:

(2) f is continuous if and only if for every objects v, w, $\tilde{e}$ such that v, $w \in \operatorname{dom} f$ and $\tilde{e}$ joins $f(v)$ and $f(w)$ in G_2 there exists an object e such that e joins v and w in G_1.

(3) f is continuous if and only if for every vertices v, w of G_1 such that v, $w \in \operatorname{dom} f$ holds v and w are adjacent iff $f_{/v}$ and $f_{/w}$ are adjacent.

Let G_1, G_2 be graphs. One can check that every partial vertex mapping from G_1 to G_2 which is directed-continuous is also continuous and every partial vertex mapping from G_1 to G_2 which is empty is also one-to-one, directed-continuous, directed, and continuous and every partial vertex mapping from G_1 to G_2 which is total is also non empty and every partial vertex mapping from G_1 to G_2 which is onto is also non empty.

Let G_1 be a simple graph and G_2 be a graph. Observe that every partial vertex mapping from G_1 to G_2 which is directed-continuous is also directed.

Let G_1 be a graph and G_2 be a simple graph. Observe that every partial vertex mapping from G_1 to G_2 which is directed and continuous is also directed-continuous.

Let G_1 be a trivial graph and G_2 be a graph. Let us observe that every partial vertex mapping from G_1 to G_2 is directed and every partial vertex mapping from G_1 to G_2 which is continuous is also directed-continuous and every partial vertex mapping from G_1 to G_2 which is non empty is also total.

Let G_1 be a graph and G_2 be a trivial graph. One can verify that every partial vertex mapping from G_1 to G_2 which is non empty is also onto.

Let G_2 be a trivial, loopless graph. Let us note that every partial vertex mapping from G_1 to G_2 is directed-continuous and continuous.

Let G_1, G_2 be graphs. Observe that there exists a partial vertex mapping from G_1 to G_2 which is empty, one-to-one, directed, continuous, and directed-continuous.

Now we state the proposition:

(4) Let us consider graphs G_1, G_2, and a partial function f from the vertices of G_1 to the vertices of G_2. Then f is a directed partial vertex mapping from G_1 to G_2 if and only if for every objects v, w, e such that $v, w \in \operatorname{dom} f$ and e joins v to w in G_1 there exists an object $\tilde{e}$ such that $\tilde{e}$ joins $f(v)$ to $f(w)$ in G_2. The theorem is a consequence of (1).

Let G_1 be a loopless graph and G_2 be a graph. One can verify that there exists a partial vertex mapping from G_1 to G_2 which is non empty, one-to-one, and directed.

Let G_1, G_2 be loopless graphs. Let us observe that there exists a partial vertex mapping from G_1 to G_2 which is non empty, one-to-one, directed, continuous, and directed-continuous.

Let G_1, G_2 be non loopless graphs. One can verify that there exists a partial vertex mapping from G_1 to G_2 which is non empty, one-to-one, directed, continuous, and directed-continuous.

Now we state the propositions:

(5) Let us consider a graph G. Then id_α is a directed, continuous, directed-continuous partial vertex mapping from G to G, where α is the vertices of G. The theorem is a consequence of (1) and (2).

(6) Let us consider graphs G_1, G_2, and a partial vertex mapping f from G_1 to G_2. Suppose f is total. Then

(i) if G_2 is loopless, then G_1 is loopless, and

(ii) if G_2 is edgeless, then G_1 is edgeless.

The theorem is a consequence of (1).

(7) Let us consider graphs G_1, G_2, and a continuous partial vertex mapping f from G_1 to G_2. Suppose f is onto. Then

(i) if G_1 is loopless, then G_2 is loopless, and

(ii) if G_1 is edgeless, then G_2 is edgeless.

The theorem is a consequence of (2).

Let G_1, G_2 be graphs and f be a partial vertex mapping from G_1 to G_2. We say that f is isomorphism if and only if

(Def. 5) f is total, one-to-one, and onto and for every vertices v, w of G_1, $\overline{\overline{G_1.\mathrm{edgesBetween}(\{v\},\{w\})}} = \overline{\overline{G_2.\mathrm{edgesBetween}(\{f(v)\},\{f(w)\})}}$.

We say that f is directed-isomorphism if and only if

(Def. 6) f is total, one-to-one, and onto and for every vertices v, w of G_1, $\overline{\overline{G_1.\mathrm{edgesDBetween}(\{v\},\{w\})}} = \overline{\overline{G_2.\mathrm{edgesDBetween}(\{f(v)\},\{f(w)\})}}$ and $\overline{\overline{G_1.\mathrm{edgesDBetween}(\{w\},\{v\})}} = \overline{\overline{G_2.\mathrm{edgesDBetween}(\{f(w)\},\{f(v)\})}}$.

Let us note that every partial vertex mapping from G_1 to G_2 which is isomorphism is also total, one-to-one, onto, and continuous and every partial vertex mapping from G_1 to G_2 which is directed-isomorphism is also total, one-to-one, onto, isomorphism, continuous, directed, and directed-continuous.

Now we state the proposition:

(8) Let us consider non-multi graphs G_1, G_2, and a partial vertex mapping f from G_1 to G_2. Suppose f is total, one-to-one, and continuous. Let us consider vertices v, w of G_1. Then $\overline{\overline{G_1.\mathrm{edgesBetween}(\{v\},\{w\})}} = \overline{\overline{G_2.\mathrm{edgesBetween}(\{f(v)\},\{f(w)\})}}$. The theorem is a consequence of (2) and (1).

Let G_1, G_2 be non-multi graphs and f be a partial vertex mapping from G_1 to G_2. Note that f is isomorphism if and only if the condition (Def. 7) is satisfied.

(Def. 7) f is total, one-to-one, onto, and continuous.

Observe that every partial vertex mapping from G_1 to G_2 which is total, one-to-one, onto, and continuous is also isomorphism.

Now we state the proposition:

(9) Let us consider non-directed-multi graphs G_1, G_2, and a partial vertex mapping f from G_1 to G_2. Suppose f is total, one-to-one, directed, and directed-continuous. Let us consider vertices v, w of G_1. Then

(i) $\overline{\overline{G_1.\mathrm{edgesDBetween}(\{v\},\{w\})}} = \overline{\overline{G_2.\mathrm{edgesDBetween}(\{f(v)\},\{f(w)\})}}$, and

(ii) $\overline{\overline{G_1.\mathrm{edgesDBetween}(\{w\},\{v\})}} = \overline{\overline{G_2.\mathrm{edgesDBetween}(\{f(w)\},\{f(v)\})}}$.

Let G_1, G_2 be non-directed-multi graphs and f be a partial vertex mapping from G_1 to G_2. Observe that f is directed-isomorphism if and only if the condition (Def. 8) is satisfied.

(Def. 8) f is total, one-to-one, onto, directed, and directed-continuous.

One can check that every partial vertex mapping from G_1 to G_2 which is total, one-to-one, onto, directed, and directed-continuous is also directed-isomorphism.

Let G be a graph. Let us observe that there exists a partial vertex mapping from G to G which is directed-isomorphism and isomorphism.

Now we state the proposition:

(10) Let us consider a graph G. Then id_α is a directed-isomorphism, isomorphism partial vertex mapping from G to G, where α is the vertices of G. The theorem is a consequence of (5).

Let G_1, G_2 be graphs and f be a partial vertex mapping from G_1 to G_2. We say that f is invertible if and only if

(Def. 9) f is one-to-one and continuous.

Note that every partial vertex mapping from G_1 to G_2 which is invertible is also one-to-one and continuous and every partial vertex mapping from G_1 to G_2 which is one-to-one and continuous is also invertible and every partial vertex mapping from G_1 to G_2 which is isomorphism is also invertible and every partial vertex mapping from G_1 to G_2 which is directed-isomorphism is also invertible and there exists a partial vertex mapping from G_1 to G_2 which is empty and invertible.

Let G_1, G_2 be loopless graphs. Note that there exists a partial vertex mapping from G_1 to G_2 which is non empty, directed, and invertible.

Let G_1, G_2 be non loopless graphs. Observe that there exists a partial vertex mapping from G_1 to G_2 which is non empty, directed, and invertible.

Let G_1, G_2 be graphs and f be an invertible partial vertex mapping from G_1 to G_2. Note that the functor f^{-1} yields a partial vertex mapping from G_2 to G_1. Observe that f^{-1} is one-to-one, continuous, and invertible as a partial vertex mapping from G_2 to G_1.

Let G_1, G_2, G_3 be graphs, f be a partial vertex mapping from G_1 to G_2, and g be a partial vertex mapping from G_2 to G_3. One can check that the functor $g \cdot f$ yields a partial vertex mapping from G_1 to G_3.

Let us consider graphs G_1, G_2, G_3, a partial vertex mapping f from G_1 to G_2, and a partial vertex mapping g from G_2 to G_3. Now we state the propositions:

(11) If f is continuous and g is continuous, then $g \cdot f$ is continuous. The theorem is a consequence of (2).

(12) If f is directed and g is directed, then $g \cdot f$ is directed.

(13) If f is directed-continuous and g is directed-continuous, then $g \cdot f$ is directed-continuous.

(14) If f is isomorphism and g is isomorphism, then $g \cdot f$ is isomorphism.

(15) If f is directed-isomorphism and g is directed-isomorphism, then $g \cdot f$ is directed-isomorphism.

2. The Relation Between Graph Mappings and Vertex Mappings

Let us consider graphs G_1, G_2 and a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(16) Suppose for every vertices v, w of G_1 such that $v, w \in \mathrm{dom}(F_{\mathbb{V}})$ and v and w are adjacent there exists an object e such that $e \in \mathrm{dom}(F_{\mathbb{E}})$ and e joins v and w in G_1. Then $F_{\mathbb{V}}$ is a partial vertex mapping from G_1 to G_2.

(17) If $\mathrm{dom}(F_{\mathbb{E}}) =$ the edges of G_1, then $F_{\mathbb{V}}$ is a partial vertex mapping from G_1 to G_2. The theorem is a consequence of (16).

(18) If F is total, then $F_{\mathbb{V}}$ is a partial vertex mapping from G_1 to G_2. The theorem is a consequence of (17).

Let us consider graphs G_1, G_2 and a directed partial graph mapping F from G_1 to G_2. Now we state the propositions:

(19) Suppose for every objects v, w such that $v, w \in \mathrm{dom}(F_{\mathbb{V}})$ and there exists an object e such that e joins v to w in G_1 there exists an object e such that $e \in \mathrm{dom}(F_{\mathbb{E}})$ and e joins v to w in G_1. Then $F_{\mathbb{V}}$ is a directed partial vertex mapping from G_1 to G_2. The theorem is a consequence of (1).

(20) Suppose $\mathrm{dom}(F_{\mathbb{E}}) =$ the edges of G_1. Then $F_{\mathbb{V}}$ is a directed partial vertex mapping from G_1 to G_2. The theorem is a consequence of (19).

(21) If F is total, then $F_{\mathbb{V}}$ is a directed partial vertex mapping from G_1 to G_2. The theorem is a consequence of (20).

Let us consider graphs G_1, G_2 and a semi-continuous partial graph mapping F from G_1 to G_2. Now we state the propositions:

(22) Suppose $F_{\mathbb{V}}$ is a partial vertex mapping from G_1 to G_2 and for every vertices v, w of G_1 such that $v, w \in \mathrm{dom}(F_{\mathbb{V}})$ and $(F_{\mathbb{V}})_{/v}$ and $(F_{\mathbb{V}})_{/w}$ are adjacent there exists an object $\tilde{e}$ such that $\tilde{e} \in \mathrm{rng}\, F_{\mathbb{E}}$ and $\tilde{e}$ joins $(F_{\mathbb{V}})(v)$ and $(F_{\mathbb{V}})(w)$ in G_2. Then $F_{\mathbb{V}}$ is a continuous partial vertex mapping from G_1 to G_2. The theorem is a consequence of (2).

(23) Suppose $\mathrm{dom}(F_{\mathbb{E}}) =$ the edges of G_1 and $\mathrm{rng}\, F_{\mathbb{E}} =$ the edges of G_2. Then $F_{\mathbb{V}}$ is a continuous partial vertex mapping from G_1 to G_2. The theorem is a consequence of (17) and (22).

(24) If F is total and onto, then $F_{\mathbb{V}}$ is a continuous partial vertex mapping from G_1 to G_2. The theorem is a consequence of (23).

Let us consider graphs G_1, G_2 and a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(25) If F is isomorphism, then there exists a partial vertex mapping f from G_1 to G_2 such that $F_{\mathbb{V}} = f$ and f is isomorphism. The theorem is a consequence of (18).

(26) If F is directed-isomorphism, then there exists a directed partial vertex mapping f from G_1 to G_2 such that $F_{\mathbb{V}} = f$ and f is directed-isomorphism. The theorem is a consequence of (21).

(27) Let us consider graphs G_1, G_2, a partial vertex mapping f from G_1 to G_2, a representative selection of the parallel edges E_1 of G_1, and a representative selection of the parallel edges E_2 of G_2. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) $F_{\mathbb{V}} = f$, and

(ii) $\mathrm{dom}(F_{\mathbb{E}}) = E_1 \cap G_1$.edgesBetween$(\mathrm{dom}\, f)$, and

(iii) $\mathrm{rng}\, F_{\mathbb{E}} \subseteq E_2 \cap G_2$.edgesBetween$(\mathrm{rng}\, f)$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exist objects v, w such that v, $w \in \mathrm{dom}\, f$ and $\$_1 \in E_1$ and $\$_2 \in E_2$ and $\$_1$ joins v and w in G_1 and $\$_2$ joins $f(v)$ and $f(w)$ in G_2. For every objects e_1, e_2, e_3 such that $e_1 \in E_1 \cap G_1$.edgesBetween$(\mathrm{dom}\, f)$ and $\mathcal{P}[e_1, e_2]$ and $\mathcal{P}[e_1, e_3]$ holds $e_2 = e_3$.

For every object e_1 such that $e_1 \in E_1 \cap G_1$.edgesBetween$(\mathrm{dom}\, f)$ there exists an object e_2 such that $\mathcal{P}[e_1, e_2]$. Consider g being a function such that $\mathrm{dom}\, g = E_1 \cap G_1$.edgesBetween$(\mathrm{dom}\, f)$ and for every object e_1 such that $e_1 \in E_1 \cap G_1$.edgesBetween$(\mathrm{dom}\, f)$ holds $\mathcal{P}[e_1, g(e_1)]$. For every object y such that $y \in \mathrm{rng}\, g$ holds $y \in E_2 \cap G_2$.edgesBetween$(\mathrm{rng}\, f)$. $\square$

Let G_1, G_2 be non-multi graphs and f be a partial vertex mapping from G_1 to G_2. The functor PVM2PGM(f) yielding a partial graph mapping from G_1 to G_2 is defined by

(Def. 10) $it_{\mathbb{V}} = f$ and $\mathrm{dom}(it_{\mathbb{E}}) = G_1$.edgesBetween$(\mathrm{dom}\, f)$ and $\mathrm{rng}\, it_{\mathbb{E}} \subseteq G_2$.edgesBetween$(\mathrm{rng}\, f)$.

Now we state the proposition:

(28) Let us consider non-multi graphs G_1, G_2, and a partial vertex mapping f from G_1 to G_2. Then PVM2PGM$(f)_{\mathbb{V}} = f$.

Let G_1, G_2 be non-multi graphs and f be a partial vertex mapping from G_1 to G_2. Observe that PVM2PGM$(f)_{\mathbb{V}}$ reduces to f.

Now we state the proposition:

(29) Let us consider graphs G_1, G_2, a directed partial vertex mapping f from G_1 to G_2, a representative selection of the directed-parallel edges E_1 of G_1, and a representative selection of the directed-parallel edges E_2 of G_2. Then there exists a directed partial graph mapping F from G_1 to G_2 such that

(i) $F_{\mathbb{V}} = f$, and

(ii) $\mathrm{dom}(F_{\mathbb{E}}) = E_1 \cap G_1$.edgesBetween$(\mathrm{dom}\, f)$, and

(iii) $\mathrm{rng}\, F_{\mathbb{E}} \subseteq E_2 \cap G_2$.edgesBetween$(\mathrm{rng}\, f)$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exist objects v, w such that v, $w \in \mathrm{dom}\, f$ and $\$_1 \in E_1$ and $\$_2 \in E_2$ and $\$_1$ joins v to w in G_1 and $\$_2$ joins $f(v)$ to $f(w)$ in G_2. For every objects e_1, e_2, e_3 such that $e_1 \in E_1 \cap G_1$.edgesBetween$(\mathrm{dom}\, f)$ and $\mathcal{P}[e_1, e_2]$ and $\mathcal{P}[e_1, e_3]$ holds $e_2 = e_3$.

For every object e_1 such that $e_1 \in E_1 \cap G_1$.edgesBetween$(\mathrm{dom}\, f)$ there exists an object e_2 such that $\mathcal{P}[e_1, e_2]$. Consider g being a function such that $\mathrm{dom}\, g = E_1 \cap G_1$.edgesBetween$(\mathrm{dom}\, f)$ and for every object e_1 such that $e_1 \in E_1 \cap G_1$.edgesBetween(dom f) holds $\mathcal{P}[e_1, g(e_1)]$. For every object y such that $y \in \operatorname{rng} g$ holds $y \in E_2 \cap G_2$.edgesBetween(rng f). $\square$

Let G_1, G_2 be non-directed-multi graphs and f be a directed partial vertex mapping from G_1 to G_2. The functor DPVM2PGM(f) yielding a directed partial graph mapping from G_1 to G_2 is defined by

(Def. 11) $it_{\mathbb{V}} = f$ and $\operatorname{dom}(it_{\mathbb{E}}) = G_1$.edgesBetween(dom f) and $\operatorname{rng} it_{\mathbb{E}} \subseteq G_2$.edgesBetween(rng f).

Now we state the proposition:

(30) Let us consider non-directed-multi graphs G_1, G_2, and a directed partial vertex mapping f from G_1 to G_2. Then DPVM2PGM$(f)_{\mathbb{V}} = f$.

Let G_1, G_2 be non-directed-multi graphs and f be a directed partial vertex mapping from G_1 to G_2. One can check that DPVM2PGM$(f)_{\mathbb{V}}$ reduces to f.

Now we state the propositions:

(31) Let us consider non-multi graphs G_1, G_2, and a directed partial vertex mapping f from G_1 to G_2. Then PVM2PGM(f) = DPVM2PGM(f).

(32) Let us consider non-multi graphs G_1, G_2, and a partial vertex mapping f from G_1 to G_2. If f is total, then PVM2PGM(f) is total.

(33) Let us consider non-directed-multi graphs G_1, G_2, and a directed partial vertex mapping f from G_1 to G_2. If f is total, then DPVM2PGM(f) is total.

(34) Let us consider non-multi graphs G_1, G_2, and a partial vertex mapping f from G_1 to G_2. If f is one-to-one, then PVM2PGM(f) is one-to-one.

Proof: Set $g =$ PVM2PGM$(f)_{\mathbb{E}}$. For every objects x_1, x_2 such that x_1, $x_2 \in \operatorname{dom} g$ and $g(x_1) = g(x_2)$ holds $x_1 = x_2$. $\square$

(35) Let us consider non-directed-multi graphs G_1, G_2, and a directed partial vertex mapping f from G_1 to G_2. If f is one-to-one, then DPVM2PGM(f) is one-to-one.

Proof: Set $g =$ DPVM2PGM$(f)_{\mathbb{E}}$. For every objects x_1, x_2 such that x_1, $x_2 \in \operatorname{dom} g$ and $g(x_1) = g(x_2)$ holds $x_1 = x_2$. $\square$

(36) Let us consider non-multi graphs G_1, G_2, and a partial vertex mapping f from G_1 to G_2. If f is onto and continuous, then PVM2PGM(f) is onto.

Proof: Set $g =$ PVM2PGM$(f)_{\mathbb{E}}$. For every object e such that $e \in$ the edges of G_2 holds $e \in \operatorname{rng} g$. $\square$

(37) Let us consider non-directed-multi graphs G_1, G_2, and a directed partial vertex mapping f from G_1 to G_2. If f is onto and directed-continuous, then DPVM2PGM(f) is onto.

Proof: Set $g =$ DPVM2PGM$(f)_{\mathbb{E}}$. For every object e such that $e \in$ the edges of G_2 holds $e \in \operatorname{rng} g$. $\square$

Let us consider non-multi graphs G_1, G_2 and a partial vertex mapping f from G_1 to G_2. Now we state the propositions:

(38) If f is continuous and one-to-one, then PVM2PGM(f) is semi-continuous. The theorem is a consequence of (2) and (34).

(39) If f is continuous, then PVM2PGM(f) is continuous. The theorem is a consequence of (2).

Let us consider non-directed-multi graphs G_1, G_2 and a directed partial vertex mapping f from G_1 to G_2. Now we state the propositions:

(40) If f is one-to-one, then DPVM2PGM(f) is semi-directed-continuous and semi-continuous. The theorem is a consequence of (35).

(41) If f is directed-continuous, then DPVM2PGM(f) is directed-continuous.

(42) Let us consider non-multi graphs G_1, G_2, and a partial vertex mapping f from G_1 to G_2. If f is one-to-one, then PVM2PGM(f) is one-to-one.

(43) Let us consider non-directed-multi graphs G_1, G_2, and a directed partial vertex mapping f from G_1 to G_2. If f is one-to-one, then DPVM2PGM(f) is one-to-one.

(44) Let us consider non-multi graphs G_1, G_2, and a partial vertex mapping f from G_1 to G_2. Suppose f is total and one-to-one. Then PVM2PGM(f) is weak subgraph embedding. The theorem is a consequence of (32) and (34).

(45) Let us consider non-directed-multi graphs G_1, G_2, and a directed partial vertex mapping f from G_1 to G_2. Suppose f is total and one-to-one. Then DPVM2PGM(f) is weak subgraph embedding. The theorem is a consequence of (33) and (35).

Let us consider non-multi graphs G_1, G_2 and a partial vertex mapping f from G_1 to G_2. Now we state the propositions:

(46) If f is total, one-to-one, and continuous, then PVM2PGM(f) is strong subgraph embedding. The theorem is a consequence of (32), (34), and (39).

(47) If f is isomorphism, then PVM2PGM(f) is isomorphism. The theorem is a consequence of (32), (34), and (36).

(48) Let us consider non-directed-multi graphs G_1, G_2, and a directed partial vertex mapping f from G_1 to G_2. Suppose f is directed-isomorphism. Then DPVM2PGM(f) is directed-isomorphism. The theorem is a consequence of (33), (35), (37), and (41).

(49) Let us consider non-multi graphs G_1, G_2. Then G_2 is G_1-isomorphic if and only if there exists a partial vertex mapping f from G_1 to G_2 such that f is isomorphism. The theorem is a consequence of (25) and (47).

(50) Let us consider non-directed-multi graphs G_1, G_2. Then G_2 is G_1-directed-isomorphic if and only if there exists a directed partial vertex mapping f from G_1 to G_2 such that f is directed-isomorphism. The theorem is a consequence of (26) and (48).

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