Some Facts about Trigonometry and Euclidean Geometry

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Summary. We calculate the values of the trigonometric functions for angles: \( \frac{\pi}{3} \) and \( \frac{\pi}{6} \), by [16]. After some trigonometric identities, we demonstrate conventional trigonometric formulas in the triangle, and the geometric property, by [14], of the triangle inscribed in a semicircle, by the proposition 3.31 in [15]. Then we define the diameter of the circumscribed circle of a triangle using the definition of the area of a triangle and prove some identities of a triangle [9]. We conclude by indicating that a diameter of a circle is twice the length of the radius.

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The notation and terminology used in this paper have been introduced in the following articles: [1], [10], [11], [19], [26], [3], [12], [5], [21], [2], [25], [29], [6], [7], [24], [30], [23], [18], [27], [28], [13], and [8].

1. Values of the Trigonometric Functions for Angles: \( \frac{\pi}{3} \) and \( \frac{\pi}{6} \)

Let us consider a real number \( a \). Now we state the propositions:

1. \( \sin(\pi - a) = \sin a \).
2. \( \cos(\pi - a) = -\cos a \).
3. \( \sin(2 \cdot \pi - a) = -\sin a \).
4. \( \cos(2 \cdot \pi - a) = \cos a \).
5. \( \sin(-2 \cdot \pi + a) = \sin a \).
6. \( \cos(-2 \cdot \pi + a) = \cos a \).
(7) \( \sin\left(\frac{3\pi}{2} + a\right) = -\cos a \).
(8) \( \cos\left(\frac{3\pi}{2} + a\right) = \sin a \).
(9) \( \sin\left(\frac{3\pi}{2} + a\right) = -\sin\left(\frac{\pi}{2} - a\right) \). The theorem is a consequence of (7).
(10) \( \cos\left(\frac{3\pi}{2} + a\right) = \cos\left(\frac{\pi}{2} - a\right) \). The theorem is a consequence of (8).
(11) \( \sin\left(\frac{2\pi}{3} - a\right) = \sin\left(\frac{\pi}{3} + a\right) \).
(12) \( \cos\left(\frac{2\pi}{3} - a\right) = -\cos\left(\frac{\pi}{3} + a\right) \).
(13) \( \sin\left(\frac{2\pi}{3} + a\right) = \sin\left(\frac{\pi}{3} - a\right) \).

Now we state the propositions:
(14) \( \cos \frac{\pi}{3} = \frac{1}{2} \).
(15) \( \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \).

PROOF: \( \sin \frac{\pi}{3} \geq 0 \) by [20, (5)], [30, (79), (81)]. \( \square \)
(16) \( \tan \frac{\pi}{3} = \sqrt{3} \). The theorem is a consequence of (14) and (15).
(17) \( \sin \frac{\pi}{6} = \frac{1}{2} \). The theorem is a consequence of (14).
(18) \( \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \). The theorem is a consequence of (15).
(19) \( \tan \frac{\pi}{6} = \frac{\sqrt{3}}{3} \). The theorem is a consequence of (17) and (18).
(20) (i) \( \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2} \), and
(ii) \( \cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \), and
(iii) \( \tan\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3} \), and
(iv) \( \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} \), and
(v) \( \cos\left(-\frac{\pi}{3}\right) = \frac{1}{2} \), and
(vi) \( \tan\left(-\frac{\pi}{3}\right) = -\sqrt{3} \).
(21) (i) \( \arcsin \frac{1}{2} = \frac{\pi}{6} \), and
(ii) \( \arcsin \frac{\sqrt{3}}{2} = \frac{\pi}{3} \).

The theorem is a consequence of (15) and (17).
(22) \( \sin \frac{2\pi}{3} = \sqrt{3} \). The theorem is a consequence of (11) and (15).
(23) \( \cos \frac{2\pi}{3} = -\frac{1}{2} \). The theorem is a consequence of (12) and (14).

2. SOME TRIGONOMETRIC IDENTITIES

Now we state the proposition:
(24) Let us consider a real number \( x \). Then \( \sin(-x) = \sin x \).

Let us consider real numbers \( x, y, z \). Now we state the propositions:
(25) If \( x + y + z = \pi \), then \( \sin^2 x + \sin^2 y - 2 \cdot \sin x \cdot \sin y \cdot \cos z = \sin^2 z \).
(26) If \( x - y + z = \pi \), then \((\sin x)^2 + (\sin y)^2 + 2 \cdot \sin x \cdot \sin y \cdot \cos z = (\sin z)^2\). The theorem is a consequence of (24) and (25).

(27) Suppose \( x - (\pi - 2 \cdot \pi + y) + z = \pi \). Then \((\sin x)^2 + (\sin y)^2 + 2 \cdot \sin x \cdot \sin y \cdot \cos z = (\sin z)^2\). The theorem is a consequence of (24), (5), and (25).

(28) If \( \pi - x - (\pi - y) + z = \pi \), then \((\sin x)^2 + (\sin y)^2 + 2 \cdot \sin x \cdot \sin y \cdot \cos z = (\sin z)^2\). The theorem is a consequence of (24), (1), and (25).

Now we state the proposition:

(29) Let us consider a real number \( a \). Then \( \sin(3 \cdot a) = 4 \cdot \sin a \cdot \sin(\pi \cdot 3 + a) \cdot \sin(\pi - 3 - a) \). The theorem is a consequence of (15).

3. Trigonometric Functions and Right Triangle

Let us consider points \( A, B, C \) of \( \mathcal{R}^2 \).

Let us assume that \( A, B, C \) form a triangle. Now we state the propositions:

(30) (i) \( \angle(A, B, C) \) is not zero, and
    (ii) \( \angle(B, C, A) \) is not zero, and
    (iii) \( \angle(C, A, B) \) is not zero, and
    (iv) \( \angle(A, B, C) \) is not zero, and
    (v) \( \angle(C, B, A) \) is not zero, and
    (vi) \( \angle(B, A, C) \) is not zero.

(31) (i) \( \angle(A, B, C) = 2 \cdot \pi - \angle(C, B, A), \) and
    (ii) \( \angle(B, C, A) = 2 \cdot \pi - \angle(A, C, B), \) and
    (iii) \( \angle(C, A, B) = 2 \cdot \pi - \angle(B, A, C), \) and
    (iv) \( \angle(B, A, C) = 2 \cdot \pi - \angle(C, A, B), \) and
    (v) \( \angle(A, C, B) = 2 \cdot \pi - \angle(B, C, A), \) and
    (vi) \( \angle(C, B, A) = 2 \cdot \pi - \angle(A, B, C). \)

Now we state the proposition:

(32) Suppose \( A, B, C \) form a triangle and \(|(B - A, C - A)| = 0\). Then
    (i) \( |C - B| \cdot \sin \angle(C, B, A) = |A - C|, \) or
    (ii) \( |C - B| \cdot (-\sin \angle(C, B, A)) = |A - C|. \)

Let us assume that \( A, B, C \) form a triangle and \( \angle(B, A, C) = \frac{\pi}{2} \). Now we state the propositions:

(33) \( \angle(C, B, A) + \angle(A, C, B) = \frac{\pi}{2} \).

(34) (i) \( |C - B| \cdot \sin \angle(C, B, A) = |A - C|, \) and
    (ii) \( |C - B| \cdot \sin \angle(A, C, B) = |A - B|, \) and
Let $a$, $b$ be real numbers and $r$ be a negative real number. Let us note that circle$(a, b, r)$ is empty.

Now we state the proposition:

(36) Let us consider real numbers $a$, $b$. Then circle$(a, b, 0) = \{[a, b]\}$.

Let $a$, $b$ be real numbers. One can verify that circle$(a, b, 0)$ is trivial.

Now we state the propositions:

(37) Let us consider points $A$, $B$, $C$ of $\mathcal{E}^2_1$ and real numbers $a$, $b$, $r$. Suppose $A, B, C$ form a triangle and $A, B \in$ circle$(a, b, r)$. Then $r$ is positive. The theorem is a consequence of (36).

(38) Let us consider a point $A$ of $\mathcal{E}^2_1$, real numbers $a$, $b$, and a positive real number $r$. If $A \in$ circle$(a, b, r)$, then $A \neq [a, b]$.

(39) Let us consider points $A$, $B$, $C$ of $\mathcal{E}^2_1$ and real numbers $a$, $b$, $r$. Suppose $A, B, C$ form a triangle and $\angle(C, B, A)$, $\angle(B, A, C) \in [0, \pi]$ and $A$, $B$, $C \in$ circle$(a, b, r)$ and $[a, b] \in L(A, C)$. Then $\angle(C, B, A) = \frac{\pi}{2}$.

PROOF: Set $O = [a, b]$. Consider $J_1$ being a point of $\mathcal{E}^2_1$ such that $A = J_1$ and $|J_1 - [a, b]| = r$. Consider $J_2$ being a point of $\mathcal{E}^2_1$ such that $B = J_2$ and $|J_2 - [a, b]| = r$. Consider $J_3$ being a point of $\mathcal{E}^2_1$ such that $C = J_3$ and $|J_3 - [a, b]| = r$. $r$ is positive. $O \neq A$ and $O \neq C$. $\angle(C, B, O) < \pi$ by (26) (16), (9), [19] (47). $A, O, B$ form a triangle and $C, O, B$ form a triangle by (37), (38), [3] (72), (75)]. $\angle(C, B, O) + \angle(O, C, B) + \angle(O, B, A) + \angle(B, A, O) = \pi$ or $\angle(C, B, O) + \angle(O, C, B) + \angle(O, B, A) + \angle(B, A, O) = -\pi$ by [26] (13), [19] (47)]. $\angle(O, C, B) = \angle(C, B, O)$ and $\angle(B, A, O) = \angle(O, B, A)$. □

(40) Let us consider points $A$, $B$, $C$ of $\mathcal{E}^2_1$ and a positive real number $r$. Suppose $\angle(A, B, C)$ is not zero. Then $\sin(r \cdot \angle(A, B, C)) = \sin(r \cdot 2 \cdot \pi) \cdot \cos(r \cdot \angle(A, B, C)) - \cos(r \cdot 2 \cdot \pi) \cdot \sin(r \cdot \angle(A, B, C))$.

(41) Let us consider points $A$, $B$, $C$ of $\mathcal{E}^2_1$. Suppose $\angle(A, B, C)$ is not zero. Then $\sin \frac{\angle(C, B, A)}{3} = \frac{\sqrt{3}}{2} \cdot \cos \frac{\angle(A, B, C)}{3} + \frac{1}{2} \cdot \sin \frac{\angle(A, B, C)}{3}$. The theorem is a consequence of (40), (22), and (23).
5. Diameter of the Circumcircle of a Triangle

Let us consider points $A, B, C$ of $\mathcal{E}_T^2$. Now we state the propositions:

(42) (i) area of $\triangle (A, B, C) = $ area of $\triangle (B, C, A)$, and
(ii) area of $\triangle (A, B, C) = $ area of $\triangle (C, A, B)$.

(43) area of $\triangle (A, B, C) = -\text{(area of } \triangle (B, A, C))$.

Let $A, B, C$ be points of $\mathcal{E}_T^2$. The function $\phi_3 (A, B, C)$ yielding a real number is defined by the term

$$\frac{|A-B||B-C||C-A|}{2 \text{area } \triangle (A,B,C)}.$$ 

Let us consider points $A, B, C$ of $\mathcal{E}_T^2$.

Let us assume that $A,B,C$ form a triangle. Now we state the propositions:

(44) $\phi_3 (A, B, C) = \frac{|C-A|}{\sin \angle (C,B,A)}$.

(45) $\phi_3 (A, B, C) = -\frac{|C-A|}{\sin \angle (A,B,C)}$. The theorem is a consequence of (44).

Now we state the proposition:

(46) $\phi_3 (A, B, C) = \phi_3 (B, C, A)$.

Let us consider points $A, B, C$ of $\mathcal{E}_T^2$.

Let us assume that $A,B,C$ form a triangle. Now we state the propositions:

(47) $\phi_3 (A, B, C) = -\phi_3 (B, A, C)$. The theorem is a consequence of (43).

(48) $\phi_3 (A, B, C) = -\phi_3 (A, C, B)$. The theorem is a consequence of (42) and (47).

(49) $\phi_3 (A, B, C) = -\phi_3 (C, B, A)$. The theorem is a consequence of (48) and (42).

6. Some Identities of a Triangle

Let us consider points $A, B, C$ of $\mathcal{E}_T^2$.

Let us assume that $A,B,C$ form a triangle. Now we state the propositions:

(50) (i) $|A - B| = \phi_3 (A, B, C) \cdot \sin \angle (A, C, B)$, and
(ii) $|B - C| = \phi_3 (A, B, C) \cdot \sin \angle (B, A, C)$, and
(iii) $|C - A| = \phi_3 (A, B, C) \cdot \sin \angle (C, B, A)$.

The theorem is a consequence of (42).

(51) $|A - B| = \phi_3 (A, B, C) \cdot 4 \cdot \sin \left(\frac{\angle (A,C,B)}{3}\right) \cdot \sin \left(\frac{\pi - \angle (A,C,B)}{3}\right)$.

The theorem is a consequence of (29).

Let us consider points $A, B, C, P$ of $\mathcal{E}_T^2$. Now we state the propositions:

(52) Suppose $A, B, P$ are mutually different and $\angle (P, B, A) = \frac{\angle (C,B,A)}{3}$ and $\angle (B, A, P) = \frac{\angle (B,A,C)}{3}$ and $\angle (A, P, B) < \pi$. Then $|A - P| \cdot \sin (\pi - \left(\frac{\angle (C,B,A)}{3} + \frac{\angle (B,A,C)}{3}\right)) = |A - B| \cdot \sin \left(\frac{\angle (C,B,A)}{3}\right)$. 


(53) Suppose $A$, $B$, $P$ are mutually different and $\angle(P, B, A) = \frac{\angle(C, B, A)}{3}$ and 
$\angle(B, A, P) = \frac{\angle(B, A, C)}{3}$ and $\angle(A, P, B) \neq \frac{\angle(A, C, B)}{3}$. Then $|A - P| \cdot \sin \left(\frac{2\pi}{3} + \frac{\angle(A, C, B)}{3}\right) = |A - B| \cdot \sin \frac{\angle(C, B, A)}{3}$.

Now we state the propositions:

(54) Let us consider points $A$, $B$, $C$ of $\mathcal{E}^2_T$. Suppose $A,B,C$ form a triangle 
and $\angle(C, A, B) < \pi$. Then

(i) $\angle(C, B, A) + \angle(B, A, C) + \angle(A, C, B) = 5 \cdot \pi$, and

(ii) $\angle(C, A, B) + \angle(A, B, C) + \angle(B, C, A) = \pi$.

Let us consider points $A$, $B$, $C$, $P$ of $\mathcal{E}^2_T$. Now we state the propositions:

(55) Suppose $A,B,C$ form a triangle and $\angle(C, B, A) < \pi$ and $A$, $B$, $P$ are mutually different and $\angle(P, B, A) = \frac{\angle(C, B, A)}{3}$ and $\angle(B, A, P) = \frac{\angle(B, A, C)}{3}$ and $\angle(A, P, B) < \pi$. Then $|A - P| \cdot \sin \left(\frac{\pi}{3} - \frac{\angle(A, C, B)}{3}\right) = |A - B| \cdot \sin \frac{\angle(C, B, A)}{3}$.

The theorem is a consequence of (1).

(56) Suppose $A,B,C$ form a triangle and $A,B,P$ form a triangle and $\angle(C, B, A) < \pi$ and $\angle(A, P, B) < \pi$ and $\angle(P, B, A) = \frac{\angle(C, B, A)}{3}$ and $\angle(B, A, P) = \frac{\angle(B, A, C)}{3}$ and $\sin \left(\frac{\pi}{3} - \frac{\angle(A, C, B)}{3}\right) \neq 0$. Then $|A - P| = -\epsilon \cdot (C, B, A) \cdot 4 \cdot \sin \frac{\angle(A, C, B)}{3} \cdot \sin \frac{\angle(C, B, A)}{3}$.

The theorem is a consequence of (53), (29), (50), (13), and (49).

7. Diameter of a Circle

Now we state the propositions:

(57) Let us consider points $A$, $B$, $C$ of $\mathcal{E}^2_T$. Suppose $A$, $B$, $C$ are mutually 
different and $C \in \mathcal{L}(A, B)$. Then $|A - B| = |A - C| + |C - B|$.

(58) Let us consider points $A$, $B$ of $\mathcal{E}^2_T$, real numbers $a$, $b$, and a positive 
real number $r$. Suppose $A$, $B$, $[a, b]$ are mutually different and $A, B \in \text{circle}(a, b, r)$ and $[a, b] \in \mathcal{L}(A, B)$. Then $|A - B| = 2 \cdot r$. The theorem is a 
consequence of (57).

(59) Let us consider real numbers $a$, $b$, a positive real number $r$, and a subset 
$C$ of $\mathcal{E}^2$. If $C = \text{circle}(a, b, r)$, then $\emptyset C = 2 \cdot r$.

PROOF: For every points $x, y$ of $\mathcal{E}^2$ such that $x, y \in C$ holds $\rho(x, y) \leq 2 \cdot r$ 
by [11] (22), (67)], [17] (4)], [22] (5)]. For every real number $s$ such that for 
every points $x, y$ of $\mathcal{E}^2$ such that $x, y \in C$ holds $\rho(x, y) \leq s$ holds $2 \cdot r \leq s$ 
References


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