Definition and Properties of Direct Sum Decomposition of Groups

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Summary. In this article, direct sum decomposition of group is mainly discussed. In the second section, support of element of direct product group is defined and its properties are formalized. It is formalized here the fact that an element of direct product group belongs to its direct sum if and only if support of the element is finite. In the third section, product map and sum map are prepared. In the fourth section, internal and external direct sum are defined. In the last section, an equivalent form of internal direct sum is proved. We referred to [23], [22], [8] and [18] in the formalization.

MSC: 20E34 03B35

Keywords: group theory; direct sum decomposition

MML identifier: GROUP_19 version: 8.1.03 5.29.1227

The notation and terminology used in this paper have been introduced in the following articles: [1], [2], [6], [16], [24], [10], [11], [12], [13], [7], [27], [20], [21], [28], [29], [30], [17], [33], [25], [3], [5], [14], [19], [32], [31], and [15].

1. Miscellanies

Let $D$ be a non empty set and $x$ be an element of $D$. Observe that the functor $\langle x \rangle$ yields a finite sequence of elements of $D$. Let $I$ be a set.

\footnote{This work was supported by JSPS KAKENHI 22300285.}
A group family of $I$ is an associative, group-like multiplicative magma family of $I$. Let $G$ be a group. Note that there exists a subgroup of $G$ which is commutative.

Now we state the proposition:

(1) Let us consider a set $I$, a group family $F$ of $I$, and an object $i$. If $i \in I$, then $F(i)$ is a group.

Let $I$ be a set, $F$ be a group family of $I$, and $i$ be an object. Assume $i \in I$. One can verify that the functor $F(i)$ yields a group. One can verify that sum $F$ is non empty and constituted functions.

Now we state the propositions:

(2) Let us consider a set $I$ and a function $F$. Suppose $I = \text{dom} F$ and for every object $i$ such that $i \in I$ holds $F(i)$ is a group. Then $F$ is a group family of $I$.

(3) Let us consider a set $I$, a group family $F$ of $I$, and an element $a$ of $\prod F$.

Then $\text{dom} a = I$.

(4) Let us consider a non empty set $I$, a group family $F$ of $I$, and an element $x$ of $I$. Then $(\text{the support of $F$})(x) = \Omega_{F(x)}$.

(5) Let us consider a non empty set $I$, a group family $F$ of $I$, a function $x$, and an element $i$ of $I$. If $x \in \prod F$, then $x(i) \in F(i)$. The theorem is a consequence of (4).

(6) Let us consider groups $G$, $H$ and a subgroup $I$ of $H$. Then every homomorphism from $G$ to $I$ is a homomorphism from $G$ to $H$.

2. Support of Element of Direct Product Group

Let $I$ be a set, $F$ be a group family of $I$, and $a$ be a function. The functor $\text{support}(a, F)$ yielding a subset of $I$ is defined by

(Def. 1) For every object $i$, $i \in it$ iff $a(i) \neq 1_{F(i)}$ and $i \in I$.

Now we state the proposition:

(7) Let us consider a set $I$, a group family $F$ of $I$, and an element $a$ of sum $F$.

Then there exists a finite subset $J$ of $I$ and there exists a many sorted set $b$ indexed by $J$ such that $J = \text{support}(a, F)$ and $a = 1_{\prod F} \cdot b$ and for every object $j$ such that $j \in I \setminus J$ holds $a(j) = 1_{F(j)}$ and for every object $j$ such that $j \in J$ holds $a(j) = b(j)$.

PROOF: Consider $g$ being an element of $\prod (\text{the support of $F$})$, $J$ being a finite subset of $I$, $b$ being a many sorted set indexed by $J$ such that $g = 1_{\prod F}$ and $a = g \cdot b$ and for every set $j$ such that $j \in J$ there exists a group-like, non empty multiplicative magma $G$ such that $G = F(j)$ and
\( b(j) \in \) the carrier of \( G \) and \( b(j) \neq 1_G \). \( \text{dom} \, \prod F = I \). For every object \( j \) such that \( j \in I \setminus J \) holds \( a(j) = 1_{F(j)} \) by [13] (11), [17] (6). For every object \( j, j \in \text{support}(a, F) \) iff \( j \in J \) by [13] (13). \( \square \)

Let \( I \) be a set, \( F \) be a group family of \( I \), and \( a \) be an element of sum \( F \). One can verify that \( \text{support}(a, F) \) is finite.

Let \( G \) be a group and \( a \) be a function from \( I \) into \( G \). The functor \( \text{support} \) yielding a subset of \( I \) is defined by

(Def. 2) For every object \( i, i \in I \) iff \( a(i) \neq 1_G \) and \( i \in I \).

We say that \( a \) is finite-support if and only if

(Def. 3) \( \text{support} \) \( a \) is finite.

Let us observe that there exists a function from \( I \) into \( G \) which is finite-support.

Let \( a \) be a finite-support function from \( I \) into \( G \). One can verify that \( \text{support} \) \( a \) is finite.

The functor \( \prod a \) yielding an element of \( G \) is defined by the term

(Def. 4) \( \prod (a \restriction \text{support} \, a) \).

Now we state the propositions:

(8) Let us consider a set \( I \), a group family \( F \) of \( I \), and an element \( a \) of \( \prod F \).

Then \( a \in \text{sum} \) \( F \) if and only if \( \text{support}(a, F) \) is finite.

Proof: Reconsider \( J = \text{support}(a, F) \) as a finite subset of \( I \). Set \( k = a \restriction J \).

Set \( x = 1_{\prod F} + k \). For every set \( j \) such that \( j \in J \) there exists a group-like, non empty multiplicative magma \( G \) such that \( G = F(j) \) and \( k(j) \in \) the carrier of \( G \) and \( k(j) \neq 1_G \) by [10] (49), (5). \( \text{dom} \, x = I \). For every object \( i \) such that \( i \in \text{dom} \, x \) holds \( x(i) = a(i) \) by [13] (11), [17] (6), [13] (13), [10] (49). \( x = a. \) \( \square \)

(9) Let us consider a set \( I \), a group \( G \), a group family \( H \) of \( I \), a function \( x \) from \( I \) into \( G \), and an element \( y \) of \( \prod H \). Suppose \( x = y \) and for every object \( i \) such that \( i \in I \) holds \( H(i) \) is a subgroup of \( G \). Then \( \text{support} \, x = \text{support}(y, H) \).

Proof: For every object \( i \) such that \( i \in I \) holds \( 1_{H(i)} = 1_G \) by [28] (44). For every object \( i, i \in \text{support}(y, H) \) iff \( i \in \text{support} \, x. \) \( \square \)

(10) Let us consider a set \( I \), a group \( G \), a group family \( F \) of \( I \), and an object \( a \). Suppose \( a \in \text{sum} \) \( F \) and for every object \( i \) such that \( i \in I \) holds \( F(i) \) is a subgroup of \( G \). Then \( a \) is a finite-support function from \( I \) into \( G \).

Proof: Reconsider \( b = a \) as an element of \( \prod F \). For every object \( i \) such that \( i \in I \) holds \( \Omega_{F(i)} \subseteq \Omega_G \), \( \text{dom} \, b = I \). For every object \( z \) such that \( z \in \text{rng} \, b \) holds \( z \in \Omega_G \) by (3), (5), [28] (40). \( \text{support}(b, F) = \text{support} \, b. \) \( \square \)
(11) Let us consider a non empty set $I$ and a group family $F$ of $I$. Then $\text{support}(1 \prod_{F} F)$ is empty.

\text{Proof:} For every object $i$, $i \notin \text{support}(1 \prod_{F} F)$ by \[17\] (6). □

(12) Let us consider a non empty set $I$, a group $G$, and a function $a$ from $I$ into $G$. If $a = I \mapsto 1_{G}$, then support $a$ is empty.

\text{Proof:} For every object $i$, $i \notin \text{support} a$ by \[24\] (7). □

(13) Let us consider a non empty set $I$, a group $G$, and a group family $F$ of $I$. Suppose for every element $i$ of $I$, $F(i)$ is a subgroup of $G$. Then $1 \prod_{F} F = I \mapsto 1_{G}$.

\text{Proof:} \text{dom} 1 \prod_{F} F = I. For every object $j$ such that $j \in \text{support} 1 \prod_{F} F$ holds $1 \prod_{F} F(j) = (I \mapsto 1_{G})(j)$ by \[17\] (6), \[24\] (7), \[28\] (44). □

(14) Let us consider a non empty set $I$, a group family $F$ of $I$, a group $G$, and a finite-support function $x$ from $I$ into $G$. Suppose support $x = \emptyset$ and for every object $i$ such that $i \in I$ holds $F(i)$ is a subgroup of $G$. Then $x = 1 \prod_{F} F$.

\text{Proof:} For every set $i$ such that $i \in I$ there exists a group-like, non empty multiplicative magma $G$ such that $G = F(i)$ and $x(i) = 1_{G}$ by \[28\] (44). □

(15) Let us consider a set $I$, a group $G$, and a finite-support function $x$ from $I$ into $G$. If support $x = \emptyset$, then $\prod x = 1_{G}$.

(16) Let us consider a non empty set $I$, a group $G$, and a finite-support function $a$ from $I$ into $G$. If $a = I \mapsto 1_{G}$, then $\prod a = 1_{G}$. The theorem is a consequence of (12) and (15).

Let us consider a non empty set $I$, a group family $F$ of $I$, an element $x$ of $\prod F$, an element $i$ of $I$, and an element $g$ of $F(i)$. Now we state the propositions:

(17) If $x = 1 \prod_{F} F + \cdot (i, g)$, then support$(x, F) \subseteq \{i\}$.

\text{Proof:} For every object $j$ such that $j \in \text{support}(x, F)$ holds $j \in \{i\}$ by \[20\] (1). □

(18) If $x = 1 \prod_{F} F + \cdot (i, g)$ and $g \neq 1_{F(i)}$, then support$(x, F) = \{i\}$. The theorem is a consequence of (17).

Let us consider a non empty set $I$, a group $G$, an element $i$ of $I$, an element $g$ of $G$, and a function $a$ from $I$ into $G$. Now we state the propositions:

(19) If $a = (I \mapsto 1_{G}) + \cdot (i, g)$, then support $a \subseteq \{i\}$.

\text{Proof:} For every object $j$ such that $j \in \text{support} a$ holds $j \in \{i\}$ by \[7\] (32), \[24\] (7). □

(20) If $a = (I \mapsto 1_{G}) + \cdot (i, g)$ and $g \neq 1_{G}$, then support $a = \{i\}$. The theorem is a consequence of (19).

Now we state the propositions:
(21) Let us consider a non empty set $I$, a group $G$, a finite-support function $a$ from $I$ into $G$, an element $i$ of $I$, and an element $g$ of $G$. If $a = (I \mapsto \mathbf{1}_G) + (i, g)$, then $\prod a = g$. The theorem is a consequence of (20) and (16).

(22) Let us consider a non empty set $I$, a group family $F$ of $I$, a function $x$, an element $i$ of $I$, and an element $g$ of $F(i)$. Suppose $\text{support}(x, F)$ is finite. Then $\text{support}((x + (i, g)), F)$ is finite.

**Proof:** Reconsider $y = x + (i, g)$ as a function. For every object $j$ such that $j \in \text{support}(y, F)$ holds $j \in \text{support}(x, F) \cup \{i\}$ by \([15] (32)\). □

(23) Let us consider a non empty set $I$, a group $G$, a function $a$ from $I$ into $G$, an element $i$ of $I$, and an element $g$ of $G$. Suppose support $a$ is finite. Then support $(a + (i, g))$ is finite.

**Proof:** Reconsider $b = a + (i, g)$ as a function from $I$ into $G$. For every object $j$ such that $j \in \text{support} b$ holds $j \in \text{support} a \cup \{i\}$ by \([17] (32)\). □

Let us consider a non empty set $I$, a group family $F$ of $I$, a function $x$, an element $i$ of $I$, and an element $g$ of $F(i)$. Now we state the propositions:

(24) If $x \in \prod F$, then $x + (i, g) \in \prod F$.

**Proof:** dom $x = I$. Set $y = x + (i, g)$. For every object $j$ such that $j \in \text{dom} \left(\text{the support of } F\right)$ holds $y(j) \in \left(\text{the support of } F\right)(j)$ by \([17] (31)\), \([4] \), \([17] (32)\), \([2] (9)\). □

(25) If $x \in \text{sum } F$, then $x + (i, g) \in \text{sum } F$.

**Proof:** Set $y = x + (i, g)$. $y \in \prod F$. For every object $j$ such that $j \in \text{support}(y, F)$ holds $j \in \text{support}(x, F) \cup \{i\}$ by \([17] (32)\). □

Now we state the propositions:

(26) Let us consider a non empty set $I$, a group $G$, a finite-support function $a$ from $I$ into $G$, an element $i$ of $I$, and an element $g$ of $G$. Then $a + (i, g)$ is a finite-support function from $I$ into $G$. The theorem is a consequence of (23).

(27) Let us consider a non empty set $I$, a group family $F$ of $I$, an object $i$, and functions $a$, $b$. Suppose $i \in I$ and dom $a = I$ and $b = a + (i, \mathbf{1}_F(i))$. Then support $(b, F) = \text{support}(a, F) \setminus \{i\}$.

**Proof:** For every object $j$, $j \in \text{support}(b, F)$ iff $j \in \text{support}(a, F) \setminus \{i\}$ by \([15] (11), (48)\), \([17] (32)\), \([15] (50)\). □

(28) Let us consider a non empty set $I$, a group $G$, an object $i$, and functions $a$, $b$ from $I$ into $G$. Suppose $i \in \text{support } a$ and $b = a + (i, \mathbf{1}_G)$. Then support $b = \text{support } a \setminus \{i\}$.

**Proof:** For every object $j$, $j \in \text{support } b$ iff $j \in \text{support } a \setminus \{i\}$ by \([15] (11), (48)\), \([17] (32)\), \([15] (50)\). □

(29) Let us consider a non empty set $I$, a group family $F$ of $I$, an object $i$, an element $a$ of sum $F$, and a function $b$. Suppose $i \in \text{support}(a, F)$ and
$$b = a \cdot (i, 1_{F(i)})$$. Then support(b, F) = support(a, F) − 1. The theorem is a consequence of (3) and (27).

(30) Let us consider a non empty set $I$, a group $G$, an object $i$, a finite-support function $a$ from $I$ into $G$, and a function $b$ from $I$ into $G$. Suppose $i \in \text{support}(a)$ and $b = a \cdot (i, 1_G)$. Then support($b$, $F$) = support($a$, $F$) − 1. The theorem is a consequence of (28).

Let us consider a non empty set $I$, a group family $F$ of $I$, and elements $a, b$ of $\prod F$.

Let us assume that support($a$, $F$) misses support($b$, $F$). Now we state the propositions:

(31) $a \cdot b \mid \text{support}(b, F) = a \cdot b$.  
**Proof**: Reconsider $c = a \cdot b \mid \text{support}(b, F)$ as a function. Reconsider $d = a \cdot b$ as an element of $\prod F$. dom $a = I$. dom $b = I$. dom $d = I$. For every object $i$ such that $i \in I$ holds $c(i) = d(i)$ by (5), [13, (11)], [17, (1)], [13, (13)]. □

(32) $a \cdot b = b \cdot a$.

**Proof**: Reconsider $c = a \cdot b$ as an element of $\prod F$. Reconsider $d = b \cdot a$ as an element of $\prod F$. dom $c = I$. dom $d = I$. For every object $i$ such that $i \in I$ holds $c(i) = d(i)$ by (5), [17, (1)]. □

Now we state the propositions:

(33) Let us consider a non empty set $I$, a group family $F$ of $I$, and an element $i$ of $I$. Then ProjGroup($F, i$) is a subgroup of sum $F$.

**Proof**: Set $S = \text{ProjGroup}(F, i)$. Set $G = \text{sum} F$. For every object $x$ such that $x \in \Omega_S$ holds $x \in \Omega_G$ by [28, (40)], (17), (8). □

(34) Let us consider a non empty set $I$, group families $F, G$ of $I$, and functions $x, y$. Suppose for every object $i$ such that $i \in I$ there exists a homomorphism $h_1$ from $F(i)$ to $G(i)$ such that $y(i) = h_1(x(i))$. Then support($y, G$) $\subseteq$ support($x, F$).

**Proof**: For every object $i$ such that $i \in \text{support}(y, G)$ holds $i \in \text{support}(x, F)$ by [30, (31)]. □

3. Product Map and Sum Map

Let $F, G$ be non-empty, non empty functions and $h$ be a non empty function. Assume dom $F = \text{dom} G = \text{dom} h$ and for every object $i$ such that $i \in \text{dom} h$ holds $h(i)$ is a function from $F(i)$ into $G(i)$. The functor $\text{ProductMap}(F, G, h)$ yielding a function from $\prod F$ into $\prod G$ is defined by
(Def. 5) For every element $x$ of $\prod F$ and for every object $i$ such that $i \in \text{dom } h$ there exists a function $h_1$ from $F(i)$ into $G(i)$ such that $h_1 = h(i)$ and $(i \iota(x))(i) = h_1((x(i)))$.

Let us consider non-empty, non empty functions $F$, $G$ and a non empty function $h$.

Let us assume that $\text{dom } F = \text{dom } G = \text{dom } h$ and for every object $i$ such that $i \in \text{dom } h$ there exists a function $h_1$ from $F(i)$ into $G(i)$ such that $h_1 = h(i)$ and $h_1$ is onto. Now we state the propositions:

(35) ProductMap($F, G, h$) is onto.

**Proof:** Set $p = \text{ProductMap}(F, G, h)$. For every object $i$ such that $i \in \text{dom } h$ holds $h(i)$ is a function from $F(i)$ into $G(i)$. For every object $y$ such that $y \in \prod G$ there exists an object $x$ such that $x \in \prod F$ and $y = p(x)$ by [2, (9)], [11, (11)], [10, (2)]. □

(36) ProductMap($F, G, h$) is one-to-one.

**Proof:** Set $p = \text{ProductMap}(F, G, h)$. For every object $i$ such that $i \in \text{dom } h$ holds $h(i)$ is a function from $F(i)$ into $G(i)$. For every objects $x_1, x_2$ such that $x_1, x_2 \in \prod F$ and $p(x_1) = p(x_2)$ holds $x_1 = x_2$ by [2] (9)], [11] (19], [10] (2)]. □

(37) ProductMap($F, G, h$) is bijective. The theorem is a consequence of (35) and (36).

Now we state the proposition:

(38) Let us consider a non empty set $I$, group families $F$, $G$ of $I$, a non empty function $h$, an element $x$ of $\prod F$, and an element $y$ of $\prod G$. Suppose $I = \text{dom } h$ and $y = (\text{ProductMap}((\text{the support of } F), (\text{the support of } G), h))(x)$ and for every object $i$ such that $i \in I$ holds $h(i)$ is a homomorphism from $F(i)$ to $G(i)$. Let us consider an object $i$. Suppose $i \in I$. Then there exists a homomorphism $h_1$ from $F(i)$ to $G(i)$ such that

(i) $h_1 = h(i)$, and

(ii) $y(i) = h_1(x(i))$.

The theorem is a consequence of (4).

Let $I$ be a non empty set, $F$, $G$ be group families of $I$, and $h$ be a non empty function. Assume $I = \text{dom } h$ and for every object $i$ such that $i \in I$ holds $h(i)$ is a homomorphism from $F(i)$ to $G(i)$. The functor $\text{ProductMap}(F, G, h)$ yielding a homomorphism from $\prod F$ to $\prod G$ is defined by the term

(Def. 6) $\text{ProductMap}((\text{the support of } F), (\text{the support of } G), h)$.

Now we state the propositions:

(39) Let us consider a non empty set $I$, group families $F$, $G$ of $I$, a non empty function $h$, an element $x$ of $\prod F$, and an element $y$ of $\prod G$. Suppose
$I = \text{dom} \ h$ and $y = (\text{ProductMap}(F, G, h))(x)$ and for every object $i$ such that $i \in I$ holds $h(i)$ is a homomorphism from $F(i)$ to $G(i)$. Let us consider an object $i$. Suppose $i \in I$. Then there exists a homomorphism $h_1$ from $F(i)$ to $G(i)$ such that

(i) $h_1 = h(i)$, and
(ii) $y(i) = h_1(x(i))$.

The theorem is a consequence of (38).

(40) Let us consider a non empty set $I$, group families $F$, $G$ of $I$, and a non empty function $h$. Suppose $I = \text{dom} \ h$ and for every object $i$ such that $i \in I$ there exists a homomorphism $h_1$ from $F(i)$ to $G(i)$ such that $h_1 = h(i)$ and $h_1$ is bijective. Then $\text{ProductMap}(F, G, h)$ is bijective. The theorem is a consequence of (4) and (37).

Let $I$ be a non empty set, $F$ be a group family of $I$, and $i$ be an element of $I$. Observe that the functor $\text{ProjGroup}(F, i)$ yields a strict subgroup of sum $F$. Let $F$, $G$ be group families of $I$ and $h$ be a non empty function. Assume $I = \text{dom} \ h$ and for every object $i$ such that $i \in I$ holds $h(i)$ is a homomorphism from $F(i)$ to $G(i)$. The functor $\text{SumMap}(F, G, h)$ yielding a homomorphism from sum $F$ to sum $G$ is defined by the term

(Def. 7) $\text{ProductMap}(F, G, h) \upharpoonright \text{sum} \ F$.

Now we state the propositions:

(41) Let us consider a non empty set $I$, group families $F$, $G$ of $I$, and a non empty function $h$. Suppose $I = \text{dom} \ h$ and for every object $i$ such that $i \in I$ there exists a homomorphism $h_1$ from $F(i)$ to $G(i)$ such that $h_1 = h(i)$ and $h_1$ is bijective. Then $\text{SumMap}(F, G, h)$ is bijective.

\textbf{Proof:} For every object $i$ such that $i \in I$ holds $h(i)$ is a homomorphism from $F(i)$ to $G(i)$. Set $p = \text{ProductMap}(F, G, h)$. Set $s = \text{SumMap}(F, G, h)$. $p$ is bijective. For every object $y$ such that $y \in \Omega_{\text{sum} \ G}$ holds $y \in \text{rng} \ s$ by [28 (40)], [30 (61)], (39), [30 (62)]. □

(42) Let us consider a non empty set $I$, group families $F$, $G$ of $I$, and a non empty function $h$. Suppose $I = \text{dom} \ h$ and for every object $i$ such that $i \in I$ holds $h(i)$ is a homomorphism from $F(i)$ to $G(i)$. Let us consider an element $i$ of $I$, an element $f$ of $F(i)$, and a homomorphism $h_1$ from $F(i)$ to $G(i)$. Suppose $h_1 = h(i)$. Then $(\text{SumMap}(F, G, h))((1\text{ProdHom}(F, i))(f)) = (1\text{ProdHom}(G, i))(h_1(f))$.

\textbf{Proof:} Set $x = (1\text{ProdHom}(F, i))(f)$. Set $y = (\text{SumMap}(F, G, h))(x)$. $\text{dom} \ y = I$. Consider $h_2$ being a homomorphism from $F(i)$ to $G(i)$ such that $h_2 = h(i)$ and $y(i) = h_2(x(i))$. For every element $j$ of $I$ such that $j \neq i$ holds $y(j) = 1_{G(j)}$ by [20 (1)], (39), [30 (31)]. □
4. Definition of Internal and External Direct Sum Decomposition

Now we state the proposition:

\((43)\) Let us consider a non empty set \(I\), a group \(G\), and an object \(i\). Suppose \(i \in I\). Then there exists a group family \(F\) of \(I\) and there exists a homomorphism \(h\) from sum \(F\) to \(G\) such that \(h\) is bijective and \(F = (I \mapsto \{1\}_G) + \cdot (\{i\} \mapsto G)\) and for every object \(j\) such that \(j \in I\) holds \(\mathbf{1}_{F(j)} = \mathbf{1}_G\) and for every element \(x\) of sum \(F\), \(h(x) = x(i)\) and for every element \(x\) of sum \(F\), there exists a finite subset \(J\) of \(I\) and there exists a many sorted set \(a\) indexed by \(J\) such that \(J \subseteq \{i\}\) and \(J = \text{support}(x, F)\) and \(\text{support}(x, F) = \emptyset\) or \(\text{support}(x, F) = \{i\}\) and \(x = 1_{\prod F + a}\) and for every object \(j\) such that \(j \in I \setminus J\) holds \(x(j) = 1_{F(j)}\) and for every object \(j\) such that \(j \in J\) holds \(x(j) = a(j)\).

**Proof:** Set \(v = I \mapsto \{1\}_G\). Set \(w = \{i\} \mapsto G\). Set \(F = v + \cdot w\). For every object \(j\) such that \(j \in I \setminus \{i\}\) holds \(F(j) = \{1\}_G\) by \([24]\ (7)\). For every object \(j\) such that \(j \in I\) holds \(F(j) = 1\) by \([23]\ (44)\). Define \(P\) [element of sum \(F\), element of \(G\)] \(\equiv \mathbb{S}_2 = \mathbb{S}_1(i)\). For every element \(x\) of sum \(F\), there exists an element \(y\) of \(G\) such that \(P[x, y]\) by \([28]\ (40)\), \([5]\), \([24]\ (13), (7)\). Consider \(h\) being a function from sum \(F\) into \(G\) such that for every element \(x\) of sum \(F\), \(P[x, h(x)]\) from \([11]\ Sch. 3\). For every object \(y\) such that \(y \in \Omega_G\) there exists an object \(x\) such that \(x = \Omega_{\text{sum } F}\) and \(y = h(x)\) by \([24]\ (7)\), \([24]\), \([17]\), \([8]\). For every element \(x\) of sum \(F\), \(\text{support}(x, F) \subseteq \{i\}\) by \([28]\ (40)\), \([5]\), \([28]\ (44)\). For every objects \(x_1, x_2\) such that \(x_1, x_2 \in \Omega_{\text{sum } F}\) and \(h(x_1) = h(x_2)\) holds \(x_1 = x_2\) by \([28]\ (40)\), \([3]\), \([7]\), \([10]\ (2)\). For every elements \(x, y\) of sum \(F\), \(h(x \cdot y) = h(x) \cdot h(y)\) by \([28]\ (40)\), \([5]\), \([28]\ (43)\), \([17]\ (1)\). For every element \(x\) of sum \(F\), there exists a finite subset \(J\) of \(I\) and there exists a many sorted set \(a\) indexed by \(J\) such that \(J \subseteq \{i\}\) and \(J = \text{support}(x, F)\) and \(\text{support}(x, F) = \emptyset\) or \(\text{support}(x, F) = \{i\}\) and \(x = 1_{\prod F + a}\) and for every object \(j\) such that \(j \in I \setminus J\) holds \(x(j) = 1_{F(j)}\) and for every object \(j\) such that \(j \in J\) holds \(x(j) = a(j)\) by \([15]\ (31)\), \([7]\).

\[\]

Let \(I\) be a non empty set and \(G\) be a group.

\[A\ \text{direct sum components of } G \text{ and } I\]

is a group family of \(I\) and is defined by

\((\text{Def. } 8)\) There exists a homomorphism \(h\) from sum \(it\) to \(G\) such that \(h\) is bijective.

Let \(F\) be a direct sum components of \(G\) and \(I\). We say that \(F\) is internal if and only if

\((\text{Def. } 9)\) for every element \(i\) of \(I\), \(F(i)\) is a subgroup of \(G\) and there exists a homomorphism \(h\) from sum \(F\) to \(G\) such that \(h\) is bijective and for every
finite-support function $x$ from $I$ into $G$ such that $x \in \sum F$ holds $h(x) = \prod x$.

One can verify that there exists a direct sum components of $G$ and $I$ which is internal.

5. Equivalent Expression of Internal Direct Sum Decomposition

Now we state the propositions:

(44) Let us consider a group $G$ and a non empty subset $A$ of $G$. Suppose for every elements $x$, $y$ of $G$ such that $x$, $y \in A$ holds $x \cdot y = y \cdot x$. Then $\text{gr}(A)$ is commutative.

**Proof:** For every elements $x$, $y$ of $G$ and for every elements $i$, $j$ of $Z$ such that $x$, $y \in A$ holds $x^i \cdot y^j = y^j \cdot x^i$ by [27] (39)]. For every element $y$ of $G$ and for every element $j$ of $Z$ such that $y \in A$ for every finite sequence $F$ of elements of $G$ for every finite sequence $I$ of elements of $Z$ such that $\text{len } F = \text{len } I$ and $\text{rng } F \subseteq A$ holds $\prod F^I \cdot y^j = y^j \cdot \prod F^I$ by [29] (21), (8), [32] (70)], [6] (4)]. For every elements $x$, $g$ of $G$ and for every element $i$ of $Z$ such that $x \in \text{gr}(A)$ and $g \in A$ holds $x \cdot g^i = g^i \cdot x$ by [29] (28). For every element $x$ of $G$ such that $x \in \text{gr}(A)$ for every finite sequence $F$ of elements of $G$ for every finite sequence $I$ of elements of $Z$ such that $\text{len } F = \text{len } I$ and $\text{rng } F \subseteq A$ holds $\prod F^I \cdot x = x \cdot \prod F^I$ by [29] (21), (8), [32] (70)], [6] (4)]. For every elements $x$, $y$ of $\text{gr}(A)$, $x \cdot y = y \cdot x$ by [28] (41)], [29] (28)], [28] (43)]. □

(45) Let us consider a group $G$, a subgroup $H$ of $G$, a finite sequence $a$ of elements of $G$, and a finite sequence $b$ of elements of $H$. If $a = b$, then $\prod a = \prod b$.

**Proof:** Define $\mathcal{P}[$natural number$] \equiv$ for every finite sequence $a$ of elements of $G$ for every finite sequence $b$ of elements of $H$ such that $\text{len } a = \#1$ and $a = b$ holds $\prod a = \prod b$. $\mathcal{P}[0]$ by [29] (8), [28] (44)]. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$ by [6] (4), (17)], [26] (55)], [29] (6)]. For every natural number $k$, $\mathcal{P}[k]$ from [4] Sch. 2]. □

(46) Let us consider a group $G$, an element $h$ of $G$, and a finite sequence $F$ of elements of $G$. Suppose for every natural number $k$ such that $k \in \text{dom } F$ holds $h \cdot F_k = F_k \cdot h$. Then $h \cdot \prod F = \prod F \cdot h$.

**Proof:** Define $\mathcal{P}[$natural number$] \equiv$ for every finite sequence $F$ of elements of $G$ such that $\text{len } F = \#1$ and for every natural number $i$ such that $i \in \text{dom } F$ holds $h \cdot F_i = F_i \cdot h$ holds $h \cdot \prod F = \prod F \cdot h$. $\mathcal{P}[0]$ by [29] (8). For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$ by [6] (4), (17), (5)], [14] (80)]. For every natural number $i$, $\mathcal{P}[i]$ from [4] Sch. 2]. □
47. Let us consider a group $G$ and finite sequences $F$, $F_1$, $F_2$ of elements of $G$. Suppose $\operatorname{len} F = \operatorname{len} F_1$ and $\operatorname{len} F = \operatorname{len} F_2$ and for every natural numbers $i$, $j$ such that $i$, $j \in \operatorname{dom} F$ and $i \neq j$ holds $F_{1i} \cdot F_{2j} = F_{2j} \cdot F_{1i}$ and for every natural number $k$ such that $k \in \operatorname{dom} F$ holds $F(k) = F_{1k} \cdot F_{2k}$. Then $\prod F = \prod F_1 \cdot \prod F_2$.

Proof: Define $\mathcal{P}[$natural number$] \equiv$ for every finite sequences $F$, $F_1$, $F_2$ of elements of $G$ such that $\operatorname{len} F = \$_1$ and $\operatorname{len} F = \operatorname{len} F_1$ and $\operatorname{len} F = \operatorname{len} F_2$ and for every natural numbers $i$, $j$ such that $i$, $j \in \operatorname{dom} F$ and $i \neq j$ holds $F_{1i} \cdot F_{2j} = F_{2j} \cdot F_{1i}$ and for every natural number $k$ such that $k \in \operatorname{dom} F$ holds $F(k) = F_{1k} \cdot F_{2k}$ holds $\prod F = \prod F_1 \cdot \prod F_2$. $\mathcal{P}[0]$ by [29] (8). For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [6] (4), (17), (5), [14] (80)]. For every natural number $i$, $\mathcal{P}[i]$ from [4] Sch. 2. □

48. Let us consider a group $G$ and a finite sequence $a$ of elements of $G$. Suppose for every object $i$ such that $i \in \operatorname{dom} a$ holds $a(i) = 1_G$. Then $\prod a = 1_G$.

Proof: Set $n = \operatorname{len} a$. $a = n \mapsto 1_G$ by [24] (13), [9] (57), [10] (2)]. □

49. Let us consider a finite set $I$, a group $G$, and a (the carrier of $G$)-valued, total, $I$-defined function $a$. Suppose for every object $i$ such that $i \in I$ holds $a(i) = 1_G$. Then $\prod a = 1_G$.

Proof: Set $c_1 = \operatorname{CFS}(I)$. Set $a_2 = a \cdot c_1$. For every object $i$ such that $i \in \operatorname{dom} a_2$ holds $a_2(i) = 1_G$ by [32] (27)], [10] (3), (12)]. □

50. Let us consider a finite set $A$, a non empty set $B$, and a $B$-valued, total, $A$-defined function $f$. Then $f \cdot \operatorname{CFS}(A)$ is a finite sequence of elements of $B$.

Let us consider a non empty set $I$, a group $G$, a finite-support function $a$ from $I$ into $G$, and a finite subset $W$ of $I$. Now we state the propositions:

51. If support $a \subseteq W$ and for every elements $i$, $j$ of $I$, $a(i) \cdot a(j) = a(j) \cdot a(i)$, then $\prod a = \prod |a| W$.

Proof: Reconsider $r = \operatorname{rng} a$ as a non empty subset of $G$. For every elements $x$, $y$ of $G$ such that $x$, $y \in r$ holds $x \cdot y = y \cdot x$ by [11] (113)]. □

52. Suppose support $a \subseteq W$. Then there exists a finite-support function $a_1$ from $W$ into $G$ such that

(i) $a_1 = a|W$, and

(ii) support $a = \text{support } a_1$, and

(iii) $\prod a = \prod a_1$.

Proof: For every object $i$, $i \in \text{support } a$ iff $i \in \text{support}(a|W)$ by [10] (49)]. □

Now we state the propositions:
Let us consider a non empty set \( I \), a group \( G \), a group family \( F \) of \( I \), elements \( s_1, s_2 \) of sum \( F \), and finite-support functions \( x, y, x_3 \) from \( I \) into \( G \). Suppose for every element \( i \) of \( I \), \( F(i) \) is a subgroup of \( G \) and for every elements \( i, j \) of \( I \) and for every elements \( g_1, g_2 \) of \( G \) such that \( i \neq j \) and \( g_1 \in F(i) \) and \( g_2 \in F(j) \) holds \( g_1 \cdot g_2 = g_2 \cdot g_1 \) and \( s_1 = x \) and \( s_2 = y \) and \( s_1 \cdot s_2 = x_3 \). Then \( \prod x_3 = \prod x \cdot \prod y \).

Proof: Reconsider \( W = \text{support } x \cup \text{support } y \) as a finite subset of \( I \). For every object \( i \) such that \( i \in \text{support } x_3 \) holds \( i \in W \) by (5), [28] (40), (43), [17] (1)]. For every function \( a \) from \( I \) into \( G \) and for every elements \( i, j \) of \( I \) such that \( a \in \prod F \) holds \( a(i) \cdot a(j) = a(j) \cdot a(i) \). \( \prod x = \prod(x|W) \). \( \prod y = \prod(y|W) \). \( \prod x_3 = \prod(x_3|W) \). Set \( c_1 = \text{CFS}(W) \). Reconsider \( w_1 = (x|W) \cdot c_1 \) as a finite sequence of elements of \( G \). Reconsider \( w_3 = (y|W) \cdot c_1 \) as a finite sequence of elements of \( G \). Reconsider \( w_2 = (x_3|W) \cdot c_1 \) as a finite sequence of elements of \( G \). For every natural numbers \( i, j \) such that \( i, j \in \text{dom } w_2 \) and \( i \neq j \) holds \( w_{1i} \cdot w_{3j} = w_{3j} \cdot w_{1i} \) by \( \prod \) (3), (12), (49)], (5). For every natural number \( i \) such that \( i \in \text{dom } w_2 \) holds \( w_2(i) = w_{1i} \cdot w_{3i} \) by \( \prod \) (3), (12), (49)], (5). \( \prod w_2 = \prod w_1 \cdot \prod w_3 \).

Let us consider a non empty set \( I \), a group \( G \), and a group family \( F \) of \( I \). Then \( F \) is an internal direct sum components of \( G \) and \( I \) if and only if for every element \( i \) of \( I \), \( F(i) \) is a subgroup of \( G \) and for every elements \( i, j \) of \( I \) and for every elements \( g_1, g_2 \) of \( G \) such that \( i \neq j \) and \( g_1 \in F(i) \) and \( g_2 \in F(j) \) holds \( g_1 \cdot g_2 = g_2 \cdot g_1 \) and for every element \( y \) of \( G \), there exists a finite-support function \( x \) from \( I \) into \( G \) such that \( x \in \text{sum } F \) and \( y = \prod x \) and for every finite-support functions \( x_1, x_2 \) from \( I \) into \( G \) such that \( x_1, x_2 \in \text{sum } F \) and \( \prod x_1 = \prod x_2 \) holds \( x_1 = x_2 \).

Proof: Define \( \mathcal{P}[\text{object, object}] \equiv \) there exists a finite-support function \( x \) from \( I \) into \( G \) such that \( x_1 = x \) and \( x_2 = \prod x \). For every element \( x \) of \( \text{sum } F \), there exists an element \( y \) of \( G \) such that \( \mathcal{P}[x, y] \). Consider \( h \) being a function from \( \text{sum } F \) into \( G \) such that for every element \( x \) of \( \text{sum } F \), \( \mathcal{P}[x, h(x)] \) from \( \prod \cdot \text{Sch. } 3 \). For every object \( y \) such that \( y \in \Omega_G \) there exists an object \( x \) such that \( x \in \Omega_{\text{sum } F} \) and \( y = h(x) \). For every objects \( x_1, x_2 \) such that \( x_1, x_2 \in \Omega_{\text{sum } F} \) and \( h(x_1) = h(x_2) \) holds \( x_1 = x_2 \). For every finite-support function \( a \) from \( I \) into \( G \) such that \( a \in \text{sum } F \) holds \( h(a) = \prod a \). For every elements \( x, y \) of \( \text{sum } F \), \( h(x \cdot y) = h(x) \cdot h(y) \).

References

Definition and properties of direct sum decomposition of ...