Differentiability of Polynomials over Reals

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Summary. In this article, we formalize in the Mizar system \[3\] the notion of the derivative of polynomials over the field of real numbers \[4\]. To define it, we use the derivative of functions between reals and reals \[9\].

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1. Preliminaries

From now on \(c\) denotes a complex, \(r\) denotes a real number, \(m, n\) denote natural numbers, and \(f\) denotes a complex-valued function.

Now we state the propositions:

(1) \(0 + f = f\).

(2) \(f - 0 = f\).

Let \(f\) be a complex-valued function. Observe that \(0 + f\) reduces to \(f\) and \(f - 0\) reduces to \(f\).

Now we state the propositions:

(3) \(c + f = (\text{dom } f \mapsto c) + f\).

(4) \(f - c = f - (\text{dom } f \mapsto c)\).

(5) \(c \cdot f = (\text{dom } f \mapsto c) \cdot f\).

(6) \(f + (\text{dom } f \mapsto 0) = f\). The theorem is a consequence of (3).

(7) \(f - (\text{dom } f \mapsto 0) = f\). The theorem is a consequence of (4).
(8) □^0 = \mathbb{R} \mapsto 1.

Proof: Reconsider \( s = 1 \) as an element of \( \mathbb{R} \). □^0 = \mathbb{R} \mapsto s by [8, (34)], [10, (7)]. □

2. DIFFERENTIABILITY OF REAL FUNCTIONS

One can check that every function from \( \mathbb{R} \) into \( \mathbb{R} \) which is differentiable is also continuous.

Let \( f \) be a differentiable function from \( \mathbb{R} \) into \( \mathbb{R} \). The functor \( f' \), yielding a function from \( \mathbb{R} \) into \( \mathbb{R} \), is defined by the term

(Def. 1) \( f'|_{\mathbb{R}} \).

Now we state the propositions:

(9) Let us consider a function \( f \) from \( \mathbb{R} \) into \( \mathbb{R} \). Then \( f \) is differentiable if and only if for every \( r \), \( f \) is differentiable in \( r \).

(10) Let us consider a differentiable function \( f \) from \( \mathbb{R} \) into \( \mathbb{R} \). Then \( f'(r) = f'(r) \).

Let \( f \) be a function from \( \mathbb{R} \) into \( \mathbb{R} \). Observe that \( f \) is differentiable if and only if the condition (Def. 2) is satisfied.

(Def. 2) for every \( r \), \( f \) is differentiable in \( r \).

Let us note that every function from \( \mathbb{R} \) into \( \mathbb{R} \) which is constant is also differentiable.

Now we state the proposition:

(11) Let us consider a constant function \( f \) from \( \mathbb{R} \) into \( \mathbb{R} \). Then \( f' = \mathbb{R} \mapsto 0 \).

Proof: Reconsider \( z = 0 \) as an element of \( \mathbb{R} \). \( f' = \mathbb{R} \mapsto z \) by [9, (22)], [10, (7)]. □

One can verify that \( \text{id}_\mathbb{R} \) is differentiable as a function from \( \mathbb{R} \) into \( \mathbb{R} \).

Now we state the proposition:

(12) \( \text{id}'_\mathbb{R} = \mathbb{R} \mapsto 1 \).

Proof: Set \( f = \text{id}_\mathbb{R} \). Reconsider \( z = 1 \) as an element of \( \mathbb{R} \). \( f' = \mathbb{R} \mapsto z \) by [9, (17)], [10, (7)]. □

Let us consider \( n \). One can verify that \( \Box^n \) is differentiable.

Now we state the proposition:

(13) \( \Box^n)' = n \cdot (\Box^{n-1}) \).

From now on \( f, g \) denote differentiable functions from \( \mathbb{R} \) into \( \mathbb{R} \).

\(^1\)Left-side \( f'(r) \) is the value of the derivative defined in this article for differentiable functions \( f : \mathbb{R} \mapsto \mathbb{R} \), and right-side \( f'(r) \) is the value of the derivative defined for partial functions in [9].
Let us consider $f$ and $g$. Let us observe that $f + g$ is differentiable as a function from $\mathbb{R}$ into $\mathbb{R}$ and $f - g$ is differentiable as a function from $\mathbb{R}$ into $\mathbb{R}$ and $f \cdot g$ is differentiable as a function from $\mathbb{R}$ into $\mathbb{R}$.

Let us consider $r$. One can verify that $r + f$ is differentiable as a function from $\mathbb{R}$ into $\mathbb{R}$ and $r \cdot f$ is differentiable as a function from $\mathbb{R}$ into $\mathbb{R}$ and $f - r$ is differentiable as a function from $\mathbb{R}$ into $\mathbb{R}$ and $-f$ is differentiable as a function from $\mathbb{R}$ into $\mathbb{R}$ and $f^2$ is differentiable as a function from $\mathbb{R}$ into $\mathbb{R}$.

Now we state the propositions:

(14) $(f + g)' = f' + g'$. The theorem is a consequence of (9) and (10).

(15) $(f - g)' = f' - g'$. The theorem is a consequence of (9) and (10).

(16) $(f \cdot g)' = g \cdot f' + f \cdot g'$. The theorem is a consequence of (9) and (10).

(17) $(r + f)' = f'$. The theorem is a consequence of (11), (3), (14), and (6).

(18) $(f - r)' = f'$. The theorem is a consequence of (11), (4), (15), and (7).

(19) $(r \cdot f)' = r \cdot f'$. The theorem is a consequence of (9) and (10).

(20) $(-f)' = -f'$.

3. Polynomials

In the sequel $L$ denotes a non empty zero structure and $x$ denotes an element of $L$.

Now we state the proposition:

(21) Let us consider a (the carrier of $L$)-valued function $f$, and an object $a$. Then Support$(f + \cdot (a, x)) \subseteq$ Support $f \cup \{a\}$.

Proof: $a = z$ or $z \in$ Support $f$ by [2, (32), (30)]. □

Let us consider $L$ and $x$. Let $f$ be a finite-Support sequence of $L$ and $a$ be an object. Observe that $f + \cdot (a, x)$ is finite-Support as a sequence of $L$.

Now we state the proposition:

(22) Let us consider a polynomial $p$ over $L$. If $p \neq 0, L$, then $\text{len } p - ' 1 = \text{len } p - 1$.

Let $L$ be a non empty zero structure and $x$ be an element of $L$. Let us note that $\langle x \rangle$ is constant and $\langle x, 0_L \rangle$ is constant.

Now we state the proposition:

(23) Let us consider a non empty zero structure $L$, and a constant polynomial $p$ over $L$. Then

(i) $p = 0, L$, or

(ii) $p = \langle p(0) \rangle$. 
Let us consider $L$, $x$, and $n$. The functor $\text{seq}(n, x)$ yielding a sequence of $L$ is defined by the term

\[(\text{Def. } 3) \quad 0. L + \cdot (n, x).\]

Observe that $\text{seq}(n, x)$ is finite-Support.

Now we state the propositions:

(24) $(\text{seq}(n, x))(n) = x$.

(25) If $m \neq n$, then $(\text{seq}(n, x))(m) = 0_L$.

(26) the length of $\text{seq}(n, x)$ is at most $n + 1$.

(27) If $x \neq 0_L$, then $\text{len seq}(n, x) = n + 1$.

\textbf{Proof:} Set $p = \text{seq}(n, x)$. For every $m$ such that the length of $p$ is at most $m$ holds $n + 1 \leq m$ by (24), [13] (13)]. □

(28) $\text{seq}(n, 0_L) = 0. L$. The theorem is a consequence of (24).

(29) Let us consider a right zeroed, non empty additive loop structure $L$, and elements $x, y$ of $L$. Then $\text{seq}(n, x) + \text{seq}(n, y) = \text{seq}(n, x + y)$. The theorem is a consequence of (24) and (25).

(30) Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure $L$, and an element $x$ of $L$. Then $-\text{seq}(n, x) = \text{seq}(n, -x)$. The theorem is a consequence of (24) and (25).

(31) Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure $L$, and elements $x, y$ of $L$. Then $\text{seq}(n, x) - \text{seq}(n, y) = \text{seq}(n, x - y)$. The theorem is a consequence of (30) and (29).

Let $L$ be a non empty zero structure and $p$ be a sequence of $L$. Let us consider $n$. The functor $p \upharpoonright n$ yielding a sequence of $L$ is defined by the term

\[(\text{Def. } 4) \quad p + \cdot (n, 0_L).\]

Let $p$ be a polynomial over $L$. Let us note that $p \upharpoonright n$ is finite-Support.

Let us consider a non empty zero structure $L$ and a sequence $p$ of $L$. Now we state the propositions:

(32) $(p \upharpoonright n)(n) = 0_L$.

(33) If $m \neq n$, then $(p \upharpoonright n)(m) = p(m)$.

Now we state the proposition:

(34) Let us consider a non empty zero structure $L$. Then $0. L \upharpoonright n = 0. L$. The theorem is a consequence of (32).

Let $L$ be a non empty zero structure. Let us consider $n$. One can verify that $0. L \upharpoonright n$ reduces to $0. L$.

Let us consider a non empty zero structure $L$ and a polynomial $p$ over $L$. Now we state the propositions:
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\[ (35) \text{ If } n > \text{len } p' - 1, \text{ then } p \upharpoonright n = p. \text{ The theorem is a consequence of (32).} \]

\[ (36) \text{ If } p \neq 0, L, \text{ then } \text{len}(p \upharpoonright (\text{len } p' - 1)) < \text{len } p. \]

**Proof:** Set \( m = \text{len } p' - 1. m = \text{len } p - 1. \) the length of \( p \upharpoonright m \) is at most \( \text{len } p \) by [2, (32)], [7, (8)]. \( \square \)

Now we state the proposition:

\[ (37) \text{ Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure } L, \text{ and a polynomial } p \text{ over } L. \text{ Then } p \upharpoonright (\text{len } p' - 1) + \text{Leading-Monomial } p = p. \text{ The theorem is a consequence of (32).} \]

Let \( L \) be a non empty zero structure and \( p \) be a constant polynomial over \( L. \) Observe that \( \text{Leading-Monomial } p \) is constant.

Now we state the proposition:

\[ (38) \text{ Let us consider an add-associative, right zeroed, right complementable, distributive, unital, non empty double loop structure } L, \text{ and elements } x, y \text{ of } L. \text{ Then eval(seq}(n, x), y) = (\text{seq}(n, x))(n) \cdot \text{power}(y, n). \text{ The theorem is a consequence of (28), (27), and (25).} \]

4. Differentiability of Polynomials over Reals

In the sequel \( p, q \) denote polynomials over \( \mathbb{R}_F. \)

Now we state the propositions:

\[ (39) \text{ Let us consider an element } r \text{ of } \mathbb{R}_F. \text{ Then } \text{power}(r, n) = r^n. \]

**Proof:** Define \( P[\text{natural number}] \equiv \text{power}(r, 1) = r^1. \) For every natural number \( n, P[n] \) from [1, Sch. 2]. \( \square \)

\[ (40) \Box^n = \text{FPower}(1_{\mathbb{R}_F}, n). \]

**Proof:** Reconsider \( f = \text{FPower}(1_{\mathbb{R}_F}, n) \) as a function from \( \mathbb{R} \) into \( \mathbb{R}. \)

\( \square^n = f \) by [5, (36)], (39). \( \square \)

Let us consider an element \( r \text{ of } \mathbb{R}_F. \) Now we state the propositions:

\[ (41) \text{FPower}(r, n + 1) = \text{FPower}(r, n) \cdot \text{id}_\mathbb{R}. \]

\[ (42) \text{FPower}(r, n) \text{ is a differentiable function from } \mathbb{R} \text{ into } \mathbb{R}. \]

**Proof:** Define \( P[\text{natural number}] \equiv \text{FPower}(r, 1) \) is a differentiable function from \( \mathbb{R} \) into \( \mathbb{R}. P[0] \) by [6, (66)]. For every natural number \( n \text{ such that } P[n] \text{ holds } P[n + 1]. \) For every natural number \( n, P[n] \) from [1, Sch. 2]. \( \square \)

\[ (43) \text{power}(r, n) = (\Box^n)(r). \text{ The theorem is a consequence of (40).} \]

Let us consider \( p. \) The functor \( \mathbb{R}_F \) yielding a sequence of \( \mathbb{R}_F \) is defined by

(Def. 5) for every natural number \( n, it(n) = p(n + 1) \cdot (n + 1). \)

Note that \( p' \) is finite-Support.

Now we state the propositions:
If \( p \neq 0 \in \mathbb{R}_F \), then \( \text{len } p' = \text{len } p - 1 \).

**Proof:** Set \( x = \text{len } p - 1 \). Set \( d = p' \). the length of \( d \) is at most \( x \) by [7, (8)]. For every \( n \) such that the length of \( d \) is at most \( n \) holds \( x \leq n \) by [11, (7)], [2, (10)], [10, (21)]. \( \square \)

If \( p \neq 0 \in \mathbb{R}_F \), then \( \text{len } p = \text{len } p' + 1 \). The theorem is a consequence of (44).

Let us consider a constant polynomial \( p \) over \( \mathbb{R}_F \). Then \( p' = 0 \in \mathbb{R}_F \). The theorem is a consequence of (45).

(47) \( (p + q)' = p' + q' \).

(48) \( (-p)' = -p' \).

(49) \( (p - q)' = p' - q' \). The theorem is a consequence of (47) and (48).

(50) \( \text{Leading-Monomial } p' = 0 \in \mathbb{R}_F + \cdot (\text{len } p - ' 2, p(\text{len } p - ' 1) \cdot (\text{len } p - ' 1)) \).

**Proof:** Set \( t = \text{Leading-Monomial } p \). Set \( m = \text{len } p - ' 1 \). Set \( k = \text{len } p - ' 2 \).
Reconsider \( a = p(m) \cdot m \) as an element of \( \mathbb{R}_F \). Set \( f = z + \cdot (k, a) \cdot l' = f \) by [11, (53)], [2, (31), (32)], [10, (7)]]. \( \square \)

Let us consider elements \( r, s \) of \( \mathbb{R}_F \). Then \( \langle r, s \rangle' = \langle s \rangle \).

Let us consider \( p \). The functor \( \text{Eval}(p) \) yielding a function from \( \mathbb{R} \) into \( \mathbb{R} \) is defined by the term

(Def. 6) \( \text{Polynomial-Function}(\mathbb{R}_F, p) \).

Let us note that \( \text{Eval}(p) \) is differentiable.

Now we state the propositions:

(52) \( \text{Eval}(0, \mathbb{R}_F) = \mathbb{R} \mapsto 0 \in \mathbb{R} \) by [5, (17)], [10, (7)]. \( \square \)

(53) Let us consider an element \( r \) of \( \mathbb{R}_F \). Then \( \text{Eval}(\langle r \rangle) = \mathbb{R} \mapsto r \).

**Proof:** \( \text{Eval}(\langle r \rangle) = \mathbb{R} \mapsto r(\in \mathbb{R}) \) by [6, (37)], [10, (7)]. \( \square \)

(54) If \( p \) is constant, then \( \text{Eval}(p)' = \mathbb{R} \mapsto 0 \). The theorem is a consequence of (23), (52), and (11).

(55) \( \text{Eval}(p + q) = \text{Eval}(p) + \text{Eval}(q) \).

(56) \( \text{Eval}(-p) = -\text{Eval}(p) \).

(57) \( \text{Eval}(p - q) = \text{Eval}(p) - \text{Eval}(q) \). The theorem is a consequence of (55) and (56).

(58) \( \text{Eval}(\text{Leading-Monomial } p) = \text{FPower}(p(\text{len } p - ' 1), \text{len } p - ' 1) \).

**Proof:** Set \( l = \text{Leading-Monomial } p \). Set \( m = \text{len } p - ' 1 \). Reconsider \( f = \text{FPower}(p(m), m) \) as a function from \( \mathbb{R} \) into \( \mathbb{R} \). \( \text{Eval}(l) = f \) by [5, (22)]. \( \square \)

(59) \( \text{Eval}(\text{Leading-Monomial } p) = p(\text{len } p - ' 1) \cdot (\square^{\text{len } p - ' 1}) \).

**Proof:** Set \( l = \text{Leading-Monomial } p \). Set \( m = \text{len } p - ' 1 \). Set \( f = p(m) \cdot (\square^m) \). \( \text{Eval}(l) = f \) by (39), [8, (36)], [5, (22)]. \( \square \)
Let us consider an element $r$ of $\mathbb{R}_F$. Then $\text{Eval}(\text{seq}(n, r)) = r \cdot (\square^n)$. The theorem is a consequence of (24), (43), and (38).

$\text{Eval}(p)' = \text{Eval}(p')$.

**Proof:** Define $P[n\text{ natural number}] \equiv \text{ for every } p \text{ such that } \text{len } p \leq 1 \text{ holds } \text{Eval}(p)' = \text{Eval}(p')$. $P[0]$ by [5 (5)], (46), (52), (54). If $P[n]$, then $P[n + 1]$ by (36), [5 (3)], [1 (13)], (37). $P[n]$ from [1, Sch. 2]. □

Let us consider $p$. Let us observe that $\text{Eval}(p)'$ is differentiable.

References


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